

Rotation minimizing frames and quaternionic rectifying curves

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Abstract In this paper, certain characterizations of a Rotation minimizing frame (RMF) are studied. An RMF is obtained on the direction curve $\int N(s)ds$ using a unit quaternionic curve. Some properties of this frame are given. Also, the condition of being a quaternionic rectifying curve and the condition of being a spherical curve is expressed using this frame. Moreover, the characterization of quaternionic rectifying curves is obtained similar to the characterization of spherical curves. Finally, the properties of the quaternionic rectifying curves are given.

Keywords rectifying curve · quaternionic frame · rotation minimizing frame (RMF)

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1 Introduction

Geometrically, hyper-complex numbers are encountered in four-dimensional space. This number system found by Hamilton in 1843 was called quaternions. Quaternions were historically the first example of a hypercomplex system that arose from enterprises to find a generalization of complex numbers. The reason for attempting to generalize complex numbers is that complex numbers are represented geometrically by points in the plane, and processes on them subtend the easiest geometric transformations of the plane. It has not been possible to correspond a number system similar to complex numbers to the points of three or more dimensional space. Therefore, quaternions have emerged to cover this gap in four-dimensional space [3], [17].

Quaternions are used not only in theoretical mathematics but also in applied mathematics. For example, in calculations that involve rotations in three dimensions, in computer graphics, control theory and signal processing for representing rotations and orientations, etc. Let's talk about a few studies on this subject. The spatial quaternionic rectifying curves are

determined in $\mathbb{R}^3$, and then some characterizations are attained for these curves. Also, quaternionic rectifying curves are searched, and the characterizations of these curves are given in $\mathbb{R}^4$ [13]. New quaternionic associated curves called quaternionic principal-direction curves and quaternionic principal-donor curves. Some properties and relationships are given between Frenet vectors and curvatures of these curves. Characterizations for quaternionic helices and quaternionic slant helices by the aid of their associated curves are given for spatial quaternionic curves [18]. Quaternion and pseudo-quaternion valued functions (or, psequat valued functions) of a single real variable define a curve in four-dimensional real Euclidean space. The Serret-Frenet formulae for a three-dimensional Euclidean curve is rederived by the help of spatial quaternions [1].

A Rotation minimizing frame (RMF) is put forward by Bishop as an option to the Serret-Frenet moving frame throughout a curve α in n -dimensional Euclidean space $\mathbb{E}^n$. The Serret-Frenet frame is an orthonormal frame that can be indicated for curves in $\mathbb{E}^n$. So, Serret-Frenet and Rotation minimizing frames (RMFs) are orthonormal frames. An RMF throughout a curve $\mu=\mu(s)$ in $\mathbb{E}^n$ is indicated by the tangent vector and $(n-1)$ normal vector fields N_i so that $N'_i(s)$ are proportional to $\mu'(s)$. Such a normal vector field throughout a curve is denominated a Rotation minimizing vector field. Any orthonormal basis $\{\mu'(s_0), N_1(s_0), \dots, N_{n-1}(s_0)\}$ at a point $\mu(s_0)$ expresses an exclusive RMF throughout the curve α . Thus, such an RMF is exclusively determined modula a rotation in $\mathbb{E}^{n-1}$, [2], [10], [11], [16], [19].

In this study, some characterizations of a Rotation minimizing frame (RMF) are studied. Firstly, a Rotation minimizing frame (RMF) is obtained on the direction curve $\int N(s)ds$ using a unit quaternionic curve. Some properties of this frame are given. Also, the condition of being a spherical curve is given with the help of this frame. Secondly, this frame is applied to a quaternionic rectifying curve. Contrary to other studies, these conditions have been shown to be simpler. Although the coefficients of rectifying curve are functions in other papers [4], [14], these coefficients of the quaternionic rectifying curve are constants in our paper, too. Moreover, interesting results and theorems are given. Finally, a relationship between the spherical curve and quaternionic rectifying curve is stated.

2 Preliminaries

A quaternion consists of a set of four ordered real numbers a, b, c, d with four units e_1, e_2, e_3 and e_4 , respectively such that: $q = ae_1 + be_2 + ce_3 + de_4$ or $q = V_q + S_q$, where the symbols $S_q = d$ (scalar part of q) and $V_q = ae_1 + be_2 + ce_3$ (vector part of q). The four units e_1, e_2, e_3 and e_4 have the following properties:

$$(*) \quad e_i \times e_i = -e_4, (e_4 = +1, 1 \leq i \leq 3),$$

$$(*) \quad e_i \times e_j = -e_j \times e_i = e_k, \quad (1 \leq i, j \leq 3),$$

[1], [8], [20].

We symbolize all the quaternions by Q . The multiplication $p = V_p + S_p$ and $q = V_q + S_q$ is given below:

$$p \times q = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q, \quad \forall p, q \in Q,$$

where the symbols " $\langle \cdot, \cdot \rangle$ " and " $\wedge$ " represent the scalar and cross products in three-dimensional Euclidean space $\mathbb{E}^3$ [1], [8], [20].

The conjugate of q is denoted by $\bar{q}$ and defined as follows:

$$\bar{q} = S_q - V_q = de_4 - ae_1 - be_2 - ce_3,$$

[1], [8], [20].

This conjugate provides the following bilinear form:

$$\begin{aligned} \langle \cdot, \cdot \rangle : Q \times Q &\rightarrow \mathbb{R} \\ (p, q) &\rightarrow h(p, q) = \frac{1}{2}(p \times \bar{q} + q \times \bar{p}) \text{ for } \mathbb{E}^4. \end{aligned}$$

This bilinear form is called quaternion inner product. The norm of q is denoted by

$$\|q\|^2 = q \times \bar{q} = \bar{q} \times q = a^2 + b^2 + c^2 + d^2, \quad \forall q \in Q.$$

In addition, if $\|q\| = 1$, then q is called a unit quaternion [1], [8], [20].

Definition 2.1 In $\mathbb{E}^4$, a normal vector field $Z = Z(t)$ upon a curve $\alpha = \alpha(t)$ is said to be slightly parallel or Rotation minimizing (RM) if the derivative $Z'(t)$ is in proportion to $\alpha'(t)$, see [10], [11].

Definition 2.2 Let $\alpha = \alpha(t)$ be any curve in $\mathbb{E}^4$. A Rotation minimizing frame throughout α (that is a parallel frame or a Bishop frame) is a moving orthonormal frame $\{T(t), N_1(t), N_2(t), N_3(t)\}$ where $T(t)$ is the tangent vector to α at the point $\alpha(t)$ and $\{N_1(t), N_2(t), N_3(t)\}$ are Rotation minimizing vector fields, see [10], [11].

If $\alpha = \alpha(t)$, $\alpha'(t) = T$ and $v(t)$ is a Rotation minimizing vector field, then $T \wedge v$ is an RM vector field. Thus, $\{T, v, T \wedge v\}$ is a Rotation minimizing frame. This type frame is defined in $\mathbb{E}^4$, see [12]. Moreover, in [12], a Rotation minimizing frame is obtained using Euler angles in $\mathbb{E}^4$. Now, we will define a Rotation minimizing frame using a new method.

3 Some characterizations for obtaining the rotation minimizing frame

The four-dimensional Euclidean space $\mathbb{E}^4$ is expressed with the space of unique quaternions. Let $I = [0, 1]$ be an interspace in real line $\mathbb{R}$ and let

$$\alpha : I \subset \mathbb{R} \rightarrow Q, \quad s \rightarrow \alpha(s) = \sum_{i=1}^4 \alpha_i(s) e_i, \quad e_4 = +1,$$

be a smooth curve α in $\mathbb{E}^4$ with nonzero curvatures $\{K, k, r - K\}$ and $\{T(s), N(s), N_2(s), N_3(s)\}$ characterises the Frenet frame of the curve $\alpha(s)$. Then, the Frenet formulae of the quaternionic curve $\alpha(s)$ are given by

$$\begin{bmatrix} T'(s) \\ N'(s) \\ N_2'(s) \\ N_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & r - K \\ 0 & 0 & -(r - K) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ N_2(s) \\ N_3(s) \end{bmatrix}, \quad (3.1)$$

where $K(s)$ is the principal curvature, $k(s)$ is the torsion and $(r - K)(s)$ is the bitorsion of $\alpha(s)$ [1].

Suppose that $\alpha = \alpha(s)$ is a quaternionic curve in $\mathbb{E}^4$. Then, Equation (3.1) is the Frenet frame of this curve. If N_2 and N_3 are rotated in the plane $Sp\{N_2, N_3\}$, then

$$\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix} = \begin{bmatrix} \cos \theta(s) & -\sin \theta(s) \\ \sin \theta(s) & \cos \theta(s) \end{bmatrix} \begin{bmatrix} N_2(s) \\ N_3(s) \end{bmatrix}, \quad (3.2)$$

where $\theta(s) = \int (r - K)(s) ds$. Calculating the derivatives of M_1 and M_2 , we get $M_1' = -k \cos \theta(s) N$ and $M_2' = -k \sin \theta(s) N$. Also, if $\tilde{k}_1 = k \cos \theta(s)$ and $\tilde{k}_2 = k \sin \theta(s)$, then $M_1' = -\tilde{k}_1 N$ and $M_2' = -\tilde{k}_2 N$. In addition, one can be written $\tilde{k}_1^2 + \tilde{k}_2^2 = k^2$ and $\theta(s) = \arctan(\frac{\tilde{k}_2}{\tilde{k}_1})$.

The formulae of the new frame can be given as

$$\begin{bmatrix} N'(s) \\ M_1'(s) \\ M_2'(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \tilde{k}_1 & \tilde{k}_2 & -K \\ -\tilde{k}_1 & 0 & 0 & 0 \\ -\tilde{k}_2 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ M_1(s) \\ M_2(s) \\ T(s) \end{bmatrix}. \quad (3.3)$$

Since the derivatives of T , M_1 and M_2 are in the same direction with γ' (i.e. in the tangential direction of γ), the frame $\{T, N, M_1, M_2\}$ is called **“Rotation Minimizing Frame (RMF)”** on the direction curve $\gamma(s) = \int N ds$. Here $\tilde{k}_1$, $\tilde{k}_2$ and K are Rotation minimizing curvatures of $\gamma(s) = \int N ds$.

At the moment, we can give the following theorems.

Theorem 3.1 *Let $\alpha : I \subset \mathbb{R} \rightarrow Q$ be a quaternionic curve and $\{T, N, N_2, N_3\}$ be the Frenet frame of α . Then, the frame $\{T, N, M_1, M_2\}$ is an RMF on the direction curve $\gamma = \int N(s)ds$.*

Proof. Since the derivatives of T , M_1 and M_2 are in the same direction with N , the frame is an RMF on the direction curve $\gamma = \int N(s)ds$. $\square$

Theorem 3.2 *Direction curve $\gamma = \int N(s)ds$ is a spherical curve if and only if $\lambda_1\tilde{k}_1 + \lambda_2\tilde{k}_2 + \lambda_3K + 1 = 0$. Here, $\tilde{k}_1$, $\tilde{k}_2$, K are Rotation minimizing curvatures and λ_1 , λ_2 , λ_3 are constants.*

Proof. Let $\gamma(s) = \int N(s)ds$ be a spherical curve. Then,

$$\gamma(s) = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 T, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

If the derivative of both sides are taken and if the necessary calculations are done, we can easily obtain $\lambda_1\tilde{k}_1 + \lambda_2\tilde{k}_2 + \lambda_3K + 1 = 0$. Conversely, let be $\lambda_1\tilde{k}_1 + \lambda_2\tilde{k}_2 + \lambda_3K + 1 = 0$. Here, $\tilde{k}_1$, $\tilde{k}_2$ and K are Rotation minimizing curvatures and λ_1 , λ_2 and λ_3 are constants. In this case, the proof is clear. That is to say, since this condition is satisfied, $\gamma(s) = \int N(s)ds$ is a spherical curve. $\square$

4 Quaternionic rectifying curves and rotation minimizing frames

In $\mathbb{E}^3$, rectifying curves are identified by Chen in [5]. The position vectors of these curves all the time lies in their rectifying planes. Their rectifying planes are stretched by the tangent and the binormal vector fields T and B of the curve. Hence, the position vector β of a rectifying curve provides the equation

$$\beta(s) = \lambda(s)T(s) + \mu(s)B(s),$$

for some differentiable functions $\lambda(s) = s + b$ and $\mu(s) = c \in \mathbb{R}$ in arclength functions s . A curve with $\kappa > 0$ is equal to a rectifying curve iff the proportion $\frac{\tau}{\kappa}$ of the curve is an inconstant linear function in arclength function s in $\mathbb{E}^3$.

In $\mathbb{E}^4$, if the position vector of a curve always lies in the orthogonal complement $N^\perp$ of its principal normal vector field N , then this curve is called a rectifying curve. Eventually,

$$N^\perp = \{W \in \mathbb{E}^4 \mid \langle W, N \rangle = 0\},$$

where $\langle, \rangle$ indicates the standard inner product in $\mathbb{E}^4$. Therefore, $N^\perp$ is a three-dimensional subspace of $\mathbb{E}^4$ and $N^\perp$ is extended by the tangent, the first binormal and the second binormal vector fields T , B_1 and B_2 , respectively. Hence, the position vector with regard to some selected origin, of a rectifying curve β in $\mathbb{E}^4$, makes available to the equation

$$\beta(s) = \lambda(s)T(s) + \mu(s)B_1(s) + \eta(s)B_2(s),$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\eta(s)$ in arclength function s . Then, rectifying curves in the way of their curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ are characterized and necessary and sufficient conditions for an arbitrary curve in $\mathbb{E}^4$ to be a rectifying are given [6], [7], [9].

Definition 4.1 *A curve $\alpha : I \rightarrow \mathbb{E}^n$ is a rectifying curve if for all $s \in I$ the orthogonal complement of $N(s)$, $N^\perp$ involves a fixed point, see [15].*

Theorem 4.2 *Let $\beta = \beta(s)$ be a unit speed quaternionic curve in $\mathbb{E}^4$ with nonzero curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$. Then β is congruent to a quaternionic rectifying curve if and only if*

$$\begin{aligned} & \frac{K(s)[r(s) - K(s)](s + c)}{k(s)} + \\ & \left[\frac{K(s)k(s) + (s + c)[K'(s)k(s) - K(s)k'(s)]}{k^2(s)[r(s) - K(s)]} \right]' = 0, \end{aligned} \quad (4.1)$$

for some $c \in \mathbb{R}$, see [13].

Now, we will give a new characterization of quaternionic rectifying curves using the RMF in $\mathbb{E}^4$.

Theorem 4.3 *Let α be a quaternionic curve in $\mathbb{E}^4$ and let $\{T, N, M_1, M_2\}$ be an RMF on the direction curve $\gamma(s) = \int N(s)ds$. Then, α is a quaternionic rectifying curve in $\mathbb{E}^4$ if and only if $A\tilde{k}_1 + B\tilde{k}_2 = (s + b)K$, where $A, B \in \mathbb{R}$, $\tilde{k}_1$, $\tilde{k}_2$ and K are Rotation minimizing curvatures.*

Proof. ($\Rightarrow$) Let $\alpha = \alpha(s)$ be a quaternionic rectifying curve in $\mathbb{E}^4$. Then,

$$\alpha(s) = \lambda_1 T + \lambda_2 M_1 + \lambda_3 M_2.$$

Taken derivatives of both sides, we get

$$T = \alpha' = \lambda_1' T + \lambda_1 T' + \lambda_2' M_1 + \lambda_2 M_1' + \lambda_3' M_2 + \lambda_3 M_2'.$$

Substituting $T' = KN$, $M_1' = -\tilde{k}_1 N$ and $M_2' = -\tilde{k}_2 N$, we obtain,

$$\lambda_1' = 1, \lambda_2' = 0, \lambda_3' = 0,$$

and

$$\lambda_1 K - \lambda_2 \tilde{k}_1 - \lambda_3 \tilde{k}_2 = 0.$$

Consequently, since $\lambda'_1 = 1$, $\lambda'_2 = 0$ and $\lambda'_3 = 0$, it is obtained that $\lambda_1 = s+b$, $\lambda_2 = A$ and $\lambda_3 = B$, $A, B \in \mathbb{R}$. Thus, $A\tilde{k}_1 + B\tilde{k}_2 = (s+b)K$.

($\Leftarrow$) Let $A\tilde{k}_1 + B\tilde{k}_2 = (s+b)K$, for $A, B \in \mathbb{R}$ and let $\tilde{k}_1$, $\tilde{k}_2$ and K are Rotation minimizing curvatures. Since

$$\frac{d}{ds}(\alpha(s) - (s+b)T - AM_1 - BM_2) = 0,$$

$\alpha(s) = (s+b)T + AM_1 + BM_2$ is a quaternionic rectifying curve in $\mathbb{E}^4$. Hence, the proof is completed. $\square$

Remark 4.1 As can be seen from Theorem 4.3, it is given a simpler condition than the condition of being a quaternionic rectifying curve according to Theorem 4.2.

Theorem 4.4 A C^4 -curve $x = x(s)$, $s \in [0, L]$, parametrized by arc length s with curvature κ and torsion τ is a spherical curve if and only if

$$(A \cos \int_0^s \tau ds + B \sin \int_0^s \tau ds) = \kappa^{-1}(s), \quad (4.2)$$

where A, B are constants. Moreover, a curve satisfying condition (4.2) lies on a sphere of radius $(A^2 + B^2)^{\frac{1}{2}}$, see [21].

Now, we will give the characterization of quaternionic rectifying curves similar to the characterization of the spherical curves.

Theorem 4.5 A C^4 -curve and a quaternionic curve $\alpha = \alpha(s)$ in $\mathbb{E}^4$, $s \in [0, L]$ parametrized by arc length s , with curvatures K, k and $(r-K)$, is a quaternionic rectifying curve if and only if

$$(A \cos \int_0^s (r-K)ds + B \sin \int_0^s (r-K)ds) = (s+b)\frac{K}{k}, \quad (4.3)$$

where A, B are constants.

Proof. Let $\alpha = \alpha(s)$ be a quaternionic rectifying curve having the curvatures K, k and $(r-K)$ in $\mathbb{E}^4$. Then, it is obtained that $A\tilde{k}_1 + B\tilde{k}_2 = (s+b)K$, where $A, B \in \mathbb{R}$. According to Theorem 4.3, since $\tilde{k}_1 = k \cos \theta(s)$, $\tilde{k}_2 = k \sin \theta(s)$ and $\theta(s) = \int (r-K)(s)ds$, we get

$$(A \cos \int_0^s (r-K)ds + B \sin \int_0^s (r-K)ds) = (s+b)\frac{K}{k},$$

where A, B are constants.

Conversely, let

$$(A \cos \int_0^s (r - K) ds + B \sin \int_0^s (r - K) ds) = (s + b) \frac{K}{k},$$

where A, B are constants. If $\int_0^s (r - K) ds = \theta$, then

$$(A \cos \theta + B \sin \theta) = (s + b) \frac{K}{k}.$$

If the necessary calculations are done, it is found

$$(Ak \cos \theta + Bk \sin \theta) = (s + b)K.$$

If $\tilde{k}_1 = k \cos \theta(s)$ and $\tilde{k}_2 = k \sin \theta(s)$, then, Theorem 4.3 is obtained. That is to say, $A\tilde{k}_1 + B\tilde{k}_2 = (s + b)K$, where $A, B \in \mathbb{R}$. This theorem shows that the curve is a quaternionic rectifying curve. $\square$

Result 4.1 *If $(r - K) = 0$, then $\theta = 0$. Since $\tilde{k}_1 = k \cos \theta(s)$ and $\tilde{k}_2 = k \sin \theta(s)$, it is $\tilde{k}_1 = k$ and $\tilde{k}_2 = 0$. Thus, $Ak = (s + b)K$. In addition, since $Ak = (s + b)K$, it is*

$$\frac{k}{K} = \frac{1}{A}(s + b).$$

Then, the curve given in Theorem 4.5 is a rectifying curve in $\mathbb{E}^3$, see [5].

Theorem 4.6 *Let $\alpha = \alpha(s)$ be a quaternionic curve having the curvatures K, k and $(r - K)$ in $\mathbb{E}^4$. And let $\beta = \beta(s)$ be a curve in $\mathbb{E}^3$ having $\tau = (r - K)$, $\kappa = \frac{k}{K(s + b)}$. Hence, $\alpha = \alpha(s)$ is any quaternionic rectifying curve in $\mathbb{E}^4$ if and only if $\beta = \beta(s)$ is any spherical curve in $\mathbb{E}^3$.*

Proof. Let $\alpha = \alpha(s)$ be a quaternionic rectifying curve in $\mathbb{E}^4$. In this case, according to Theorem 4.5, we can write

$$(A \cos \int_0^s (r - K) ds + B \sin \int_0^s (r - K) ds) = (s + b) \frac{K}{k}.$$

If $\tau = (r - K)$ and $\kappa = \frac{k}{K(s + b)}$, then,

$$A \cos \int_0^s \tau ds + B \sin \int_0^s \tau ds = \kappa^{-1}.$$

Thus, since Theorem 4.4, $\beta(s)$ is a spherical curve.

On the contrary, let $\beta(s)$ be a spherical curve having $\tau = (r - K)$, $\kappa = \frac{k}{K(s+b)}$. Then,

$$(A \cos \int_0^s \tau ds + B \sin \int_0^s \tau ds) = \kappa^{-1}(s).$$

Substituting $\tau = (r - K)$, $\kappa = \frac{k}{K(s+b)}$ and doing the necessary calculations, we get

$$(A \cos \int_0^s (r - K) ds + B \sin \int_0^s (r - K) ds) = (s + b) \frac{K}{k}.$$

As a result, it can be seen that α is a quaternionic rectifying curve. $\square$

At moment, we can write the following theorem.

Theorem 4.7 *Let $\alpha = \alpha(s)$ be any unit speed quaternionic rectifying curve with non-zero Rotation minimizing curvatures k_1 , k_2 and K in $\mathbb{E}^4$. Also, let h be a standard metric in $\mathbb{E}^4$. Then, the following statements are satisfied:*

(i) $h(\alpha(s), T) = s + b$, i.e. the curve is tangential.

(ii) The distance function, that is $\rho(s) = \|\alpha(s)\|$, satisfies

$$h(\alpha(s), \alpha(s)) = \|\alpha(s)\|^2 = (s + b)^2 + A^2 + B^2 = s^2 + as + c,$$

where $a, b, c, A, B \in \mathbb{R}$.

(iii) $h(\alpha(s), M_1) = A$, $h(\alpha(s), M_2) = B$, $A, B \in \mathbb{R}$.

Proof. Now, we show that this statements are satisfied.

(i) $h(\alpha(s), T) = (s + b)h(T, T) + Ah(M_1, T) + Bh(M_2, T)$. Since $h(T, T) = 1$ and $h(M_1, T) = h(M_2, T) = 0$, we get $h(\alpha(s), T) = (s + b)$.

(ii) The proof is obvious from the inner product $h(\alpha(s), \alpha(s)) = \|\alpha(s)\|^2 = (s + b)^2 + A^2 + B^2$, $b, A, B \in \mathbb{R}$.

(iii) Since $h(\alpha(s), M_1) = (s + b)h(T, M_1) + Ah(M_1, M_1) + Bh(M_2, M_1)$ and $h(T, M_1) = h(M_2, M_1) = 0$, $h(M_1, M_1) = 1$; we get $h(\alpha(s), M_1) = A = \text{constant}$. Also, we get

$$h(\alpha(s), M_2) = B = \text{constant}.$$

$\square$

Result 4.2 B_1 and B_2 are the first binormal and the second binormal vector fields, respectively. M_1 and M_2 are the first binormal and the second binormal vector fields, respectively, too. In this case, the rectifying curve can be defined as $\alpha(s) = (s + c)T + AB_1 + BB_2$. The coefficients A and B are constants for $\alpha(s) = (s + c)T + AB_1 + BB_2$ in this paper. But, coefficients B_1, B_2 of a rectifying curve are functions in $\mathbb{E}^4$, see [15]. Also, coefficients $B_1, B_2, \dots, B_{n-2}$ of a rectifying curve are functions in $\mathbb{E}^n$, see [4].

Conclusion

In this paper, some characterizations of a Rotation minimizing frame (RMF) are studied. An RMF is obtained on the direction curve $\int N(s)ds$ using a unit quaternionic curve. Also, the condition of being a spherical curve, and the condition of being a quaternionic rectifying curve is given with the help of this frame. The most important point in this paper is that unlike [4], [14], the coefficients of the quaternionic rectifying curve are constants. The second important point is that unlike [13], the condition of being a quaternionic rectifying curve is simpler.

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