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# On generalized Fibonacci and Lucas hybrid polynomials

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**Abstract:** In this paper, we introduce a new generalization of Fibonacci and Lucas hybrid polynomials. We investigate some basic properties of these polynomials such as recurrence relations, the generating functions, the Binet formulas, summation formulas, and a matrix representation. We derive generalized Cassini's identity and generalized Honsberger formula for generalized Fibonacci hybrid polynomials by using their matrix representation.

**Key words:** $r$-Fibonacci polynomial, $r$-Lucas polynomial, hybrid number

## 1. Introduction

The hybrid numbers have been recently introduced by Ozdemir [10] as a generalization of complex, hyperbolic and dual numbers. The set of hybrid numbers are defined as

$$\mathbb{K} = \left\{a + bi + c\epsilon + dh \mid i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\right\}, \tag{1.1}$$

where $a, b, c, d \in \mathbb{R}$.

The addition, substraction and multiplication of two hybrid numbers $k_1 = a_1 + b_1 i + c_1 \epsilon + d_1 h$ and $k_2 = a_2 + b_2 i + c_2 \epsilon + d_2 h$ are defined respectively as

$$k_1 \pm k_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\, i + (c_1 \pm c_2)\, \epsilon + (d_1 \pm d_2)\, h,$$

and

$$\begin{aligned} k_1 k_2 \quad = \quad & a_1 a_2 - b_1 b_2 + d_1 d_2 + b_1 c_2 + c_1 b_2 \\ & + (a_1 b_2 + b_1 a_2 + b_1 d_2 - d_1 b_2)\, i \\ & + (a_1 c_2 + c_1 a_2 + b_1 d_2 - d_1 b_2 + d_1 c_2 - c_1 d_2)\, \epsilon \\ & + (a_1 d_2 + d_1 a_2 + c_1 b_2 - b_1 c_2)\, h. \end{aligned}$$

Similar to the quaternion multiplication, the hybrid number multiplication is noncommutative. Thus the set of hybrid numbers form a noncommutative algebra. For more details of hybrid numbers, see Ozdemir's paper [10].

Recently, Szynal-Liana [15] introduced the Fibonacci hybrid polynomials (alias hybrinomials) as

$$FH_n(x) = F_n(x) + F_{n+1}(x)\, i + F_{n+2}(x)\, \epsilon + F_{n+3}(x)\, h, \tag{1.2}$$

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where $F_n(x)$ is the $n$th Fibonacci polynomial (see [7]) defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), n \geq 2$$

with initial values $F_0(x) = 0, F_1(x) = 1$. In [5], Kizilates defined the Horodam hybrinomials which generalize the Fibonacci hybrinomials. Several studies related to hybrid numbers with Fibonacci-like number coefficients can be found in [6, 8, 9, 12–14, 16–19], and for a recent study related to the generalized Fibonacci numbers and polynomials we refer to [2]. It is also worth noting that, in the literature there exist another type of hybrid polynomials which are related to the families of special functions such as the Laguerre and the Hermite polynomials, see [4]. We should note that our approach will be different from that polynomials.

This work has been intended as an attempt to introduce a new class of hybrid polynomials, which are so-called "$r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$". They give a natural generalization of the Fibonacci and Lucas hybrinomials. We give the generating functions, the Binet formulas, matrix representations and several basic properties of these hybrid polynomials. A relation between $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials is also given.

Now we start by recalling some basic results concerning to the $r$-Fibonacci polynomials and $r$-Lucas polynomials of type $s$. For the detailed information related to these polynomials, we refer to [1, 11].

Let $r \geq 1$ be any integer, and let $s = 1, 2, \ldots, r$. The $r$-Fibonacci polynomials $\left(U_n^{(r)}\right) := \left(U_n^{(r)}(x, y)\right)$ are defined by

$$U_{n+1}^{(r)} = xU_n^{(r)} + yU_{n-r}^{(r)}, \ n \geq r \tag{1.3}$$

with initial conditions $U_0^{(r)} = 0, U_k^{(r)} = x^{k-1}$ for $k = 1, 2, \ldots, r$. Its companion sequence, the $r$-Lucas polynomials of type $s$, $\left(V_n^{(r,s)}\right) := \left(V_n^{(r,s)}(x, y)\right)$ are defined by

$$V_{n+1}^{(r,s)} = xV_n^{(r,s)} + yV_{n-r}^{(r,s)}, \ n \geq r \tag{1.4}$$

with initial conditions $V_0^{(r,s)} = s + 1, V_k^{(r,s)} = x^k$ for $k = 1, 2, \ldots, r$. It is clear that if we take $r = 1, s = 1$, then these polynomials respectively reduce to the classical bivariate Fibonacci and Lucas polynomials, see [3]. If we take $x = y = 1$, they reduce to the $r$-Fibonacci and $r$-Lucas numbers.

The Binet formulas for $r$-Fibonacci polynomials and $r$-Lucas polynomials of type $s$ are

$$U_n^{(r)} = \sum_{k=1}^{r+1} \frac{\alpha_k^n}{(r+1)\alpha_k - rx}, \tag{1.5}$$

and

$$V_n^{(r,s)} = \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}, \tag{1.6}$$

respectively. Here $\alpha_k$ are the distinct roots of the polynomial $P(t) = t^{r+1} - xt^r - y$. For details see [1].

## 2. Main results

In this section, we give the definition of $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$. We give the generating functions, the Binet formulas, the summation formulas of these polynomials.

Also, we give a relation between $r$-Fibonacci hybrid polynomials and the $r$-Lucas hybrid polynomials of type $s$.

**Definition 2.1** *For $n \geq 0$, the $n$th $r$-Fibonacci hybrid polynomial and $r$-Lucas hybrid polynomial of type $s$ are defined respectively by the recurrence relations*

$$\mathbb{K}_{U^{(r)},n} = U_n^{(r)} + U_{n+1}^{(r)} i + U_{n+2}^{(r)} \epsilon + U_{n+3}^{(r)} h \tag{2.1}$$

*and*

$$\mathbb{K}_{V^{(r,s)},n} = V_n^{(r,s)} + V_{n+1}^{(r,s)} i + V_{n+2}^{(r,s)} \epsilon + V_{n+3}^{(r,s)} h, \tag{2.2}$$

*where $U_n^{(r)}$ is the $n$th $r$-Fibonacci polynomial and $V_n^{(r,s)}$ is the $n$th $r$-Lucas polynomial of type $s$.*

In Table, we state some special cases of $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$.

**Table** . Some special cases of $\mathbb{K}_{U^{(r)},n}$ and $\mathbb{K}_{V^{(r,s)},n}$.

| $x$ | $y$ | $r$ | $s$ | $\mathbb{K}_{U^{(r)},n}$ | $\mathbb{K}_{V^{(r,s)},n}$ |
|---|---|---|---|---|---|
| $x$ | $y$ | 1 | 1 | Bivariate Fibonacci hybrid polynomials | Bivariate Lucas hybrid polynomials [13] |
| $x$ | 1 | 1 | 1 | Fibonacci hybrid polynomials [15] | Lucas hybrid polynomials [15] |
| $2x$ | 1 | 1 | 1 | Pell hybrid polynomials [8] | Pell-Lucas hybrid polynomials [8] |
| $m$ | 1 | $p$ | $p$ | Gen. hybrid Fibonacci $p$-numbers [6] | Gen. hybrid Lucas $p$-numbers [6] |
| 1 | 1 | $p$ | $p$ | Hybrid Fibonacci $p$-numbers [6] | Hybrid Lucas $p$-numbers [6] |
| 1 | 1 | 1 | 1 | Fibonacci hybrid numbers [16] | Lucas hybrid numbers [14] |
| 2 | 1 | 1 | 1 | Pell hybrid numbers [17] | Pell-Lucas hybrid numbers [17] |
| 1 | 2 | 1 | 1 | Jacobsthal hybrid numbers [18] | Jacobsthal-Lucas hybrid numbers [18] |

We state the following lemma, which is useful to obtain the generating functions of $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$.

**Lemma 2.2** *For $n \geq r+1$, the $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$ satisfy the following relations*

$$\mathbb{K}_{U^{(r)},n} = x\mathbb{K}_{U^{(r)},n-1} + y\mathbb{K}_{U^{(r)},n-r-1} \tag{2.3}$$

*and*

$$\mathbb{K}_{V^{(r,s)},n} = x\mathbb{K}_{V^{(r,s)},n-1} + y\mathbb{K}_{V^{(r,s)},n-r-1}. \tag{2.4}$$

**Proof** By using the definition of $r$-Fibonacci polynomials, we have

$$\begin{aligned}
\mathbb{K}_{U^{(r)},n} &= U_n^{(r)} + U_{n+1}^{(r)} i + U_{n+2}^{(r)} \epsilon + U_{n+3}^{(r)} h \\
&= xU_{n-1}^{(r)} + yU_{n-r-1}^{(r)} + (xU_n^{(r)} + yU_{n-r}^{(r)})i + (xU_{n+1}^{(r)} + yU_{n-r+1}^{(r)})\epsilon + (xU_{n+2}^{(r)} + yU_{n-r+2}^{(r)})h \\
&= x(U_{n-1}^{(r)} + U_n^{(r)} i + U_{n+1}^{(r)} \epsilon + U_{n+2}^{(r)} h) + y(U_{n-r-1}^{(r)} + U_{n-r}^{(r)} i + U_{n-r+1}^{(r)} \epsilon + U_{n-r+2}^{(r)} h) \\
&= x\mathbb{K}_{U^{(r)},n-1} + y\mathbb{K}_{U^{(r)},n-r-1}.
\end{aligned}$$

Thus, we get the desired result.
The relation for the $r$-Lucas hybrid polynomials of type $s$ can be proven similarly. So we omit it here. □

**Theorem 2.3** *The generating functions for $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$ are, respectively:*

$$G(z)=\frac{\mathbb{K}_{U^{(r)},0}+\sum_{n=1}^{r}\left(\mathbb{K}_{U^{(r)},n}-x\mathbb{K}_{U^{(r)},n-1}\right)z^{n}}{1-xz-yz^{r+1}}, \tag{2.5}$$

*and*

$$H(z)=\frac{\mathbb{K}_{V^{(r,s)},0}+\sum_{n=1}^{r}\left(\mathbb{K}_{V^{(r,s)},n}-x\mathbb{K}_{V^{(r,s)},n-1}\right)z^{n}}{1-xz-yz^{r+1}}. \tag{2.6}$$

**Proof** Let

$$G(z)=\sum_{n=0}^{\infty}\mathbb{K}_{U^{(r)},n}z^{n}=\mathbb{K}_{U^{(r)},0}+\mathbb{K}_{U^{(r)},1}z+\mathbb{K}_{U^{(r)},2}z^{2}+\cdots+\mathbb{K}_{U^{(r)},n}z^{n}+\cdots.$$

From Lemma 2.2, we get

$$\begin{aligned}
&\left(1-xz-yz^{r+1}\right)G(z)\\
=&\sum_{n=0}^{\infty}\mathbb{K}_{U^{(r)},n}z^{n}-x\sum_{n=0}^{\infty}\mathbb{K}_{U^{(r)},n}z^{n+1}+y\sum_{n=0}^{\infty}\mathbb{K}_{U^{(r)},n}z^{n+r+1}\\
=&\sum_{n=0}^{\infty}\mathbb{K}_{U^{(r)},n}z^{n}-x\sum_{n=1}^{\infty}\mathbb{K}_{U^{(r)},n-1}z^{n}-y\sum_{n=r+1}^{\infty}\mathbb{K}_{U^{(r)},n-r-1}z^{n}\\
=&\sum_{n=0}^{r}\mathbb{K}_{U^{(r)},n}z^{n}-x\sum_{n=1}^{r}\mathbb{K}_{U^{(r)},n-1}z^{n}+\sum_{n=r+1}^{\infty}\left(\mathbb{K}_{U^{(r)},n}-x\mathbb{K}_{U^{(r)},n-1}-y\mathbb{K}_{U^{(r)},n-r-1}\right)z^{n}\\
=&\sum_{n=0}^{r}\mathbb{K}_{U^{(r)},n}z^{n}-x\sum_{n=1}^{r}\mathbb{K}_{U^{(r)},n-1}z^{n}\\
=&\mathbb{K}_{U^{(r)},0}+\sum_{n=1}^{r}\left(\mathbb{K}_{U^{(r)},n}-x\mathbb{K}_{U^{(r)},n-1}\right)z^{n}.
\end{aligned}$$

The generating function for $r$-Lucas hybrid polynomials of type $s$ can be proven similarly. So we omit it here. □

Next, we state the Binet formulas for $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$, and by using these formulas, we derive some properties of them.

**Theorem 2.4** *The Binet formulas for $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$ are, respectively:*

$$\mathbb{K}_{U^{(r)},n}=\sum_{k=1}^{r+1}\frac{\alpha_{k}^{*}\alpha_{k}^{n}}{(r+1)\alpha_{k}-rx} \tag{2.7}$$

*and*

$$\mathbb{K}_{V^{(r,s)},n} = \sum_{k=1}^{r+1} \alpha_k^* \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}, \tag{2.8}$$

*where* $\alpha_k^* = 1 + \alpha_k i + \alpha_k^2 \epsilon + \alpha_k^3 h$.

**Proof** By using the definitions of the sequences $\mathbb{K}_{U^{(r)},n}$, $\mathbb{K}_{V^{(r,s)},n}$ and the Binet formulas of $U_n^{(r)}$ and $V_n^{(r,s)}$, we have

$$\begin{aligned}
\mathbb{K}_{U^{(r)},n} &= U_n^{(r)} + U_{n+1}^{(r)} i + U_{n+2}^{(r)} \epsilon + U_{n+3}^{(r)} h \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n}{(r+1)\alpha_k - rx} + \sum_{k=1}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx} i \\
&\quad + \sum_{k=1}^{r+1} \frac{\alpha_k^{n+2}}{(r+1)\alpha_k - rx} \epsilon + \sum_{k=1}^{r+1} \frac{\alpha_k^{n+3}}{(r+1)\alpha_k - rx} h \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n + \alpha_k^{n+1} i + \alpha_k^{n+2} \epsilon + \alpha_k^{n+3} h}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n (1 + \alpha_k i + \alpha_k^2 \epsilon + \alpha_k^3 h)}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n}{(r+1)\alpha_k - rx} (1 + \alpha_k i + \alpha_k^2 \epsilon + \alpha_k^3 h) \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^n}{(r+1)\alpha_k - rx}.
\end{aligned}$$

For the $r$-Lucas hybrid polynomials of type $s$, we have

$$\begin{aligned}
\mathbb{K}_{V^{(r,s)},n} &= V_n^{(r,s)} + V_{n+1}^{(r,s)} i + V_{n+2}^{(r,s)} \epsilon + V_{n+3}^{(r,s)} h \\
&= \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} + \sum_{k=1}^{r+1} \alpha_k^{n+1} \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} i \\
&\quad + \sum_{k=1}^{r+1} \alpha_k^{n+2} \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \epsilon + \sum_{k=1}^{r+1} \alpha_k^{n+3} \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} h \\
&= \sum_{k=1}^{r+1} \frac{(\alpha_k^n + \alpha_k^{n+1} i + \alpha_k^{n+2} \epsilon + \alpha_k^{n+3} h)((s+1)\alpha_k - sx)}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n (1 + \alpha_k i + \alpha_k^2 \epsilon + \alpha_k^3 h)((s+1)\alpha_k - sx)}{(r+1)\alpha_k - rx}
\end{aligned}$$

$$= \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}(1 + \alpha_k i + \alpha_k^2 \epsilon + \alpha_k^3 h)$$

$$= \sum_{k=1}^{r+1} \alpha_k^* \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}.$$

which gives the desired results. □

The link between the $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$ can be given in the following result.

**Theorem 2.5** *The $r$-Lucas hybrid polynomials of type $s$ can be expressed in term of $r$-Fibonacci hybrid polynomials as*

$$\mathbb{K}_{V^{(r,s)},n} = \mathbb{K}_{U^{(r)},n+1} + sy\mathbb{K}_{U^{(r)},n-r} \quad for \quad n \geq r+1. \tag{2.9}$$

**Proof** Using the Binet formulas of $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$, we have

$$\begin{aligned}
\mathbb{K}_{U^{(r)},n+1} + sy\mathbb{K}_{U^{(r)},n-r} &= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^{n+1}}{(r+1)\alpha_k - rx} + sy \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^{n-r}}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^{n+1} + sy\alpha_k^* \alpha_k^{n-r}}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^n (\alpha_k + sy\alpha_k^{-r})}{(r+1)\alpha_k - ra} \\
&= \sum_{k=1}^{r+1} \alpha_k^* \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \\
&= \mathbb{K}_{V^{(r,s)},n}.
\end{aligned}$$

Note that since $\alpha_k = x + y\alpha_k^{-r}$, then $\alpha_k + sy\alpha_k^{-r} = (s+1)\alpha_k - sx$. □

Next, we give some summation formulas for $\mathbb{K}_{U^{(r)},n}$ and $\mathbb{K}_{V^{(r,s)},n}$ in the following theorem.

**Theorem 2.6** *For $m \geq 0$, we have*

$$\sum_{n=0}^{m} \mathbb{K}_{U^{(r)},n} = \sum_{k=1}^{r+1} \frac{\alpha_k^*(\alpha_k^{m+1} - 1)}{(r+1)\alpha_k^2 - (r(x+1)+1)\alpha_k + rx} \tag{2.10}$$

*and*

$$\sum_{n=0}^{m} \mathbb{K}_{V^{(r,s)},n} = \sum_{k=1}^{r+1} \alpha_k^* \frac{((s+1)\alpha_k - sx)(\alpha_k^{m+1} - 1)}{(r+1)\alpha_k^2 - (r(x+1)+1)\alpha_k + rx}. \tag{2.11}$$

**Proof** Using the Binet formula of $\mathbb{K}_{U^{(r)},n}$, we get

$$\begin{aligned}\sum_{n=0}^{m}\mathbb{K}_{U^{(r)},n} &= \sum_{n=0}^{m}\sum_{k=1}^{r+1}\frac{\alpha_k^*\alpha_k^n}{(r+1)\alpha_k-rx}\\ &= \sum_{k=1}^{r+1}\frac{\alpha_k^*}{(r+1)\alpha_k-rx}\frac{\alpha_k^{m+1}-1}{\alpha_k-1}\\ &= \sum_{k=1}^{r+1}\frac{\alpha_k^*(\alpha_k^{m+1}-1)}{(r+1)\alpha_k^2-(r(x+1)+1)\alpha_k+rx}.\end{aligned}$$

And from the Binet formula of $\mathbb{K}_{V^{(r,s)},n}$, we get

$$\begin{aligned}\sum_{n=0}^{m}\mathbb{K}_{V^{(r,s)},n} &= \sum_{n=0}^{m}\sum_{k=1}^{r+1}\alpha_k^*\alpha_k^n\frac{(s+1)\alpha_k-sx}{(r+1)\alpha_k-rx}\\ &= \sum_{k=1}^{r+1}\alpha_k^*\frac{(s+1)\alpha_k-sx}{(r+1)\alpha_k-rx}\sum_{n=0}^{m}\alpha_k^n\\ &= \sum_{k=1}^{r+1}\alpha_k^*\frac{((s+1)\alpha_k-sx)(\alpha_k^{m+1}-1)}{(r+1)\alpha_k^2-(r(x+1)+1)\alpha_k+rx}.\end{aligned}$$

□

## 3. Matrix representation

In this section, we give a matrix representation of $r$-Fibonacci hybrid polynomials.

Let $Q_r := \begin{bmatrix} x & 0 & \cdots & 0 & y\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & 1 & 0\end{bmatrix}$ be a matrix of size $(r+1)\times(r+1)$. For $n\geq r$, it can be verified that

$$Q_r^n = \begin{bmatrix} U_{n+1}^{(r)} & yU_{n-r+1}^{(r)} & \cdots & yU_{n-1}^{(r)} & yU_n^{(r)}\\ U_n^{(r)} & yU_{n-r}^{(r)} & & yU_{n-2}^{(r)} & yU_{n-1}^{(r)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ U_{n-r+1}^{(r)} & yU_{n-2r+1}^{(r)} & \cdots & yU_{n-r-1}^{(r)} & yU_{n-r}^{(r)}\end{bmatrix}$$

with $U_j^{(r)}=0$, for $j=-1,-2,\ldots$.

Now let us define the matrix

$$K(n) := \begin{bmatrix} \mathbb{K}_{U^{(r)},n+1} & y\mathbb{K}_{U^{(r)},n-r+1} & \cdots & y\mathbb{K}_{U^{(r)},n-1} & y\mathbb{K}_{U^{(r)},n}\\ \mathbb{K}_{U^{(r)},n} & y\mathbb{K}_{U^{(r)},n-r} & & y\mathbb{K}_{U^{(r)},n-2} & y\mathbb{K}_{U^{(r)},n-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbb{K}_{U^{(r)},n-r+1} & y\mathbb{K}_{U^{(r)},n-2r+1} & \cdots & y\mathbb{K}_{U^{(r)},n-r-1} & y\mathbb{K}_{U^{(r)},n-r}\end{bmatrix}. \tag{3.1}$$

For any nonnegative integer $n \geq r$, we have

$$K(n) = K(0) Q_r^n. \tag{3.2}$$

By taking the determinant of both sides of the matrix equality (3.2), we get the generalized Cassini's identity for $r$-Fibonacci hybrid polynomials as

$$\det K(n) = (-1)^{nr} y^n \det K(0). \tag{3.3}$$

**Remark 3.1** *If $r = 1$ in (3.3), we get*

$$\mathbb{K}_{U^{(1)},n+1}\mathbb{K}_{U^{(1)},n-1} - \mathbb{K}^2_{U^{(1)},n} = (-1)^n \left(\mathbb{K}_{U^{(1)},1}\mathbb{K}_{U^{(1)},-1} - \mathbb{K}^2_{U^{(1)},0}\right). \tag{3.4}$$

By using the matrix identity (3.2), we get the following theorem which can be seen as a generalization of Honsberger formula.

**Theorem 3.2** *For $n, s, t \geq r$, we have*

$$\mathbb{K}_{U^{(r)},s+t} = \mathbb{K}_{U^{(r)},s}\mathbb{K}_{U^{(r)},t+1} + y\sum_{j=0}^{r-1} \mathbb{K}_{U^{(r)},s-r+j}\mathbb{K}_{U^{(r)},t-j} \tag{3.5}$$

$$\mathbb{K}_{U^{(r)},s+t} = U_s^{(r)}\mathbb{K}_{U^{(r)},t+1} + y\sum_{j=1}^{r-1} U_{s-r+j}^{(r)}\mathbb{K}_{U^{(r)},t-j}. \tag{3.6}$$

**Proof** Let $K := K(0)$, considering the matrix equalities $(KQ_r^{s+t})K = (KQ_r^s)(Q_r^t K)$ and $(KQ_r^{s+t}) = (KQ_r^s)Q_r^t$, then equating the corresponding entries, we get the desired results respectively. □

**Remark 3.3** *If $r = 1$, the identities (3.5) and (3.6) reduce to the classical bivariate Fibonacci hybrid polynomials as*

$$\mathbb{K}_{U^{(1)},s+t} = \mathbb{K}_{U^{(1)},s}\mathbb{K}_{U^{(1)},t+1} + y\mathbb{K}_{U^{(1)},s-1}\mathbb{K}_{U^{(1)},t} \tag{3.7}$$

$$\mathbb{K}_{U^{(1)},s+t} = U_s^{(1)}\mathbb{K}_{U^{(1)},t+1} + yU_{s-1}^{(1)}\mathbb{K}_{U^{(1)},t}. \tag{3.8}$$

## 4. Conclusion

In our present investigation, we have introduced $r$-Fibonacci hybrid polynomials and $r$-Lucas hybrid polynomials of type $s$ as a generalization of the $r$-Fibonacci and $r$-Lucas hybrid polynomials. We have derived several interesting properties such as the Binet formulas, the generating functions, summation formulas, a matrix representation of these polynomials. As an application of matrix method, we have derived a generalization of Cassini's and Honsberger formulas.

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