



Some results concerning B -local variable Morrey-Lorentz spaces

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Abstract

In the present study, we obtain the boundedness of B -maximal, B -singular integral operators and B -Riesz potential in the B -local variable Morrey-Lorentz spaces. Furthermore, our results are applied to the Riesz-Bessel transform of high order and B -fractional maximal operator.

Keywords γ -rearrangement · Fractional maximal function · Laplace-Bessel differential operator · Local variable Morrey-Lorentz spaces · Maximal function · Potential operator · Singular integral

Mathematics Subject Classification 42B20 · 42B25 · 42B35 · 47G10

1 Introduction

Investigations on variable function spaces have emerged in response to problems arising in differential equations with non-standard growth conditions, fluid dynamics, calculus of variations, elasticity theory [1–8]. In recent years, there has been considerable interest in these spaces, as they offer an effective framework for investigating regularity properties in PDE. These topics are considerably popular for many researchers both in terms of their applications and also because of difficulties in studies and existing compelling problems. Therefore, many results have been presented in harmonic analysis, however, there are still gaps in existing literature. In this study, we discuss the maximal, singular and potential operators which are connected with the Laplace-Bessel differential operator in B -local variable Morrey-Lorentz spaces.

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In harmonic analysis, it is very notable that studying singular integral operators with convolution type. As applications of this theory, the boundedness of these operators and their versions which are connected with the Laplace-Bessel differential operator have great significance. Such problems have been discussed in variable function spaces by several mathematicians (for example, see [3, 6, 9–19]).

The classical Morrey spaces, introduced by Morrey [20], play an important role in the study of quasi-linear elliptic partial differential equations. Various problems in harmonic analysis have also been investigated within Morrey-type spaces and their extensions, including Morrey-Lorentz, local Morrey-Lorentz and related generalized spaces [9, 17–19, 21–31]. The Lorentz-Morrey spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ have first been introduced in [28] and the corresponding local spaces $M_{p,q;\lambda}^{\text{loc}}(\mathbb{R}^n)$ have later been developed, with fundamental properties have established in [21, 30]. These spaces can be viewed as natural extensions of Lorentz spaces, since $M_{p,q;0}^{\text{loc}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. In recent years, considerable attention has been devoted to the boundedness of classical operators such as the Hardy-Littlewood maximal operator, Calderón-Zygmund operators, Riesz potentials, and the Hilbert transform in local Morrey-Lorentz spaces [22–24]. Furthermore, generalized Lorentz-Morrey spaces and their local versions have been studied in connection with embedding theorems, Sobolev-type inequalities, and regularity estimates, revealing subtle differences in their mapping properties depending on the underlying structure [19, 28, 30, 31].

The local variable Morrey-Lorentz spaces are extensions of variable Lorentz spaces and $\mathcal{M}_{p(\cdot),q(\cdot);\lambda}^{\text{loc}} = L_{p(\cdot),q(\cdot)}$ for $\lambda = 0$. The variable Lorentz spaces $L_{p(\cdot),q(\cdot)}$ have been defined and it has been proved in [13] that fractional type, singular integral and ergodic operators are bounded on these spaces. In [17], local variable Morrey-Lorentz spaces have been defined and the boundedness of Calderón-Zygmund and maximal operators have been obtained in these spaces. Also, in mentioned study, authors have applied their results to certain operators in harmonic analysis. Later, in [9], the boundedness of the Riesz potential has been obtained in local variable Morrey-Lorentz spaces and authors have given the applications of their results to fractional Marcinkiewicz operator, fractional maximal and fractional powers of some analytic semigroups in these spaces.

Under the light of above results, in this study our aim is to characterize the boundedness of B -maximal, B -singular integral operators and B -Riesz potential in B -local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. Furthermore, we apply our main results to some operators.

2 Definitions of Some Spaces and Operators

The B -maximal, B -singular integral operators and B -Riesz potential are generated by generalized translation operators which are defined as follows:

$$T^y f(x) = A_{\gamma,k} \int_0^\pi \dots \int_0^\pi f[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''] \prod_{i=1}^k \sin^{\gamma_i-1} \alpha_i d\alpha_i,$$

where $f \in C_c^\infty(\mathbb{R}^n)$, $A_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma(\frac{\gamma_i+1}{2}) [\Gamma(\frac{\gamma_i}{2})]^{-1}$, $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos \alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, and $1 \leq k \leq n$ [32, 33]. One can observe that T^γ is associated with the Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^k B_i + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad 1 \leq k \leq n,$$

due to its fundamental properties. In particular, it commutes with Δ_B , i.e., $T^\gamma(\Delta_B f) = \Delta_B(T^\gamma f)$ and it induces a convolution structure adapted to the spectral decomposition of Δ_B via the Fourier-Bessel transform. Moreover, this operator arises naturally as the radial analogue of the classical translation operator in $\mathbb{R}^n$. In view of these properties, the operator can be naturally interpreted as a generalized translation associated with the spectral structure of Δ_B [34–36].

The B -convolution, B -maximal operators and B -Riesz potential are defined respectively, as

$$\begin{aligned} (f \otimes g)(x) &= \int_{\mathbb{R}_{k,+}^n} f(y) T^\gamma g(x)(y')^\gamma dy, \\ M_\gamma f(x) &= \sup_{r>0} |B_+(0, r)|_\gamma^{-1} \int_{B_+(0, r)} T^\gamma |f(x)| (y')^\gamma dy, \quad f \in L_{1,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n), \\ I_\gamma^\alpha f(x) &= \int_{\mathbb{R}_{k,+}^n} T^\gamma |x|^{\alpha-\mathcal{Q}} f(y)(y')^\gamma dy, \quad 0 < \alpha < \mathcal{Q}, \end{aligned}$$

where $\mathbb{R}_{k,+}^n = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \leq k \leq n\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$, and $B_+(x, r) = \{y \in \mathbb{R}_{k,+}^n : |x - y| < r\}$. If $B_+(0, r) \subset \mathbb{R}_{k,+}^n$ is measurable, then

$$|B_+(0, r)|_\gamma = \int_{B_+(0, r)} (x')^\gamma dx = r^\mathcal{Q} \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})},$$

where $\mathcal{Q} = n + |\gamma|$.

The definition of B -singular integral operator is as follows:

$$T_\gamma f(x) = p.v. \int_{\mathbb{R}_{k,+}^n} \frac{\Theta(\xi)}{|y|^\mathcal{Q}} [T^\gamma f(x)](y')^\gamma dy$$

where $\xi = y/|y|$, and $\Theta(\xi)$ belongs to any function spaces on the semisphere $\mathcal{S}_{k,+} = \{x \in \mathbb{R}_{k,+}^n : |x| = 1\}$ that ensure the convergence of the principal value integral (for example, see [35, 37]), typical choices are $\Theta(\xi) \in C(\mathcal{S}_{k,+})$ or $\Theta(\xi) \in L^p(\mathcal{S}_{k,+})$ and

also satisfies the cancellation condition

$$\int_{\mathcal{S}_{k,+}} \Theta(\xi)(\xi')^\nu d\sigma(\xi) = 0. \quad (2.1)$$

B -singular integral operator is of convolution type with the kernel $K(y) = \frac{\Theta(\xi)}{|y|^Q}$ and so one can write as $T_\gamma f(x) = (K \otimes f)(x)$.

Let $\mathcal{P}(\mathbb{R}_{k,+}^n)$ denote the set of whole measurable functions $p(\cdot) : \mathbb{R}_{k,+}^n \rightarrow [1, \infty]$. These functions are referred to as variable exponent functions. We assume that

$$1 \leq p_- \leq p(x) \leq p_+ < \infty. \quad (2.2)$$

Here $p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}_{k,+}^n} p(x)$, $p_+ = \operatorname{ess\,sup}_{x \in \mathbb{R}_{k,+}^n} p(x)$.

Given $p(\cdot)$, the conjugate exponent function, denoted by $p'(\cdot)$, satisfies the following equality:

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}_{k,+}^n.$$

By $\mathbb{P}(\mathbb{R}_{k,+}^n)$, we will denote the set of variable exponents $p(\cdot)$ such that (2.2) holds and the following local log-condition is satisfied:

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad y \in \mathbb{R}_{k,+}^n, \quad (2.3)$$

being $A > 0$ independent of x, y . The following inequalities are known as decay conditions:

$$|p(x) - p(0)| \leq \frac{A_0}{|\ln|x||}, \quad |x| \leq \frac{1}{2}, \quad (2.4)$$

$$|p(x) - p_\infty| \leq \frac{A_\infty}{|\ln|x||}, \quad |x| \geq 2, \quad (2.5)$$

where $p_\infty = \lim_{x \rightarrow \infty} p(x)$. For short, $\mathcal{P}_{0,\infty}(\mathbb{R}_{k,+}^n)$ consists of bounded measurable functions which satisfy (2.4) and (2.5).

The space $L_{p(\cdot)}(\mathbb{R}^n)$ is known as the set of measurable functions f such that

$$\|f\|_{L_{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \mu > 0 : \varrho_{p(\cdot)}(f/\mu) \leq 1 \right\},$$

for $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, where

$$\varrho_{p(\cdot)} = \int_{\mathbb{R}^n \setminus (\mathbb{R}^n)_\infty} |f(x)|^{p(x)} dx + \|f\|_{L_\infty(\mathbb{R}^n)_\infty} < \infty,$$

and $(\mathbb{R}^n)_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$.

Let E be any measurable set with $|E|_\gamma = \int_E (x')^\gamma dx$, $f : \mathbb{R}_{k,+}^n \rightarrow \mathbb{R}$ be a measurable function and $t \in (0, \infty)$. Then the γ -rearrangement of f in decreasing order and average function of f_γ^* are defined, respectively, as (see [38]):

$$f_\gamma^*(t) = \inf \left\{ \eta > 0 : \left| \{x \in \mathbb{R}_{k,+}^n : |f(x)| > \eta\} \right|_\gamma \leq t \right\},$$

$$f_\gamma^{**}(t) = \frac{1}{t} \int_0^t f_\gamma^*(\tau) d\tau.$$

Lemma 1 [39] *Let $E = (E', E_n) \subset \mathbb{R}_{k,+}^n$ be any measurable set, $E_n \subset (0, \infty)$, $E' = E_1 \times \cdots \times E_{n-1} \subset \mathbb{R}^{n-1}$ and $y \in \mathbb{R}_{k,+}^n$. Then*

$$\int_E T^y g(x) (y')^\gamma dy = A_\gamma \int_{(x,0)+\tilde{E}} g \left(z', \sqrt{z_n^2 + z_{n+1}^2} \right) z_{n+1}^{\gamma-1} dz_1 dz_2 \cdots dz_n dz_{n+1}$$

holds, where $\tilde{E} = E' \times (-\ell, \ell) \times [0, \ell)$, and $\ell = \sup E_n$.

Lemma 2 [39] *Let $E \subset \mathbb{R}_{k,+}^n$ be any measurable set and $y \in \mathbb{R}_{k,+}^n$. Then,*

$$\sup_{|E|_\gamma=t} \int_E T^y |f(x)| (y')^\gamma dy = A_\gamma \int_0^t f_\gamma^*(\tau) d\tau. \quad (2.6)$$

holds.

The variable Lorentz space $L_{p(\cdot), q(\cdot), \gamma}(\mathbb{R}_{k,+}^n)$ is the space of whole measurable functions f on $\mathbb{R}_{k,+}^n$ such that $t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_\gamma^*(t) \in L_{q(\cdot)}(0, \infty)$, i.e.,

$$\mathfrak{J}_{p(\cdot), q(\cdot)}(f) = \int_0^\infty t^{\frac{q(t)}{p(t)} - 1} |f_\gamma^*(t)|^{q(t)} dt < \infty,$$

and with finite norm

$$\begin{aligned} \|f\|_{L_{p(\cdot), q(\cdot), \gamma}(\mathbb{R}_{k,+}^n)} &= \inf \left\{ \mu > 0 : \mathfrak{J}_{p(\cdot), q(\cdot)}(f/\mu) \leq 1 \right\} \\ &= \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_\gamma^*(t) \right\|_{L_{q(\cdot)}(0, \infty)}, \end{aligned}$$

where $p(\cdot), q(\cdot) \in \mathcal{P}_0(0, \infty)$ [10]. If $p(\cdot) = q(\cdot)$, then it is easy to see that $L_{p(\cdot), p(\cdot), \gamma} = L_{p(\cdot), \gamma}$ [16, Theorem 3.2].

The local variable Morrey space is the space of whole measurable functions such that

$$\begin{aligned}\|f\|_{LM_{p(\cdot),\lambda}(0,\infty)} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{\bar{p}(\rho)}} \|f\|_{L_{p(\cdot)}(0,\rho)} \\ &= \sup_{\rho>0} \inf \left\{ \mu > 0 : \int_0^\rho \left| \frac{f(x)}{\mu \rho^{\frac{\lambda}{\bar{p}(\rho)}}} \right|^{p(x)} dx \leq 1 \right\} < \infty,\end{aligned}$$

and is denoted by $LM_{p(\cdot),\lambda} \equiv LM_{p(\cdot),\lambda}(0, \infty)$, where $1 < p_- \leq p_+ < \infty$, $\lambda \in [0, 1)$, $\bar{p}(\rho) = p(0)$ for $\rho \in (0, 1)$ and $\bar{p}(\rho) = p(\infty)$ for $\rho \geq 1$.

Now, we will introduce the B -local variable Morrey-Lorentz space $\mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.

The B -local variable Morrey-Lorentz space is the space of whole measurable functions such that

$$\|f\|_{\mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)} = \sup_{\rho>0} \rho^{-\frac{\lambda}{\bar{q}(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_\gamma^*(t) \right\|_{L_{q(\cdot)}(0,\rho)} < \infty,$$

and is denoted by $\mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}} \equiv \mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$, where $0 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$ and $\lambda \in [0, 1)$. These spaces generalize variable Lorentz spaces such that $\mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n) = L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}_{k,+}^n)$ when $\lambda = 0$ [13]. Also, if $\lambda = 0$ and $p(\cdot) = q(\cdot)$, then $\mathcal{M}_{p(\cdot),q(\cdot),\gamma;0}^{\text{loc}}(\mathbb{R}_{k,+}^n) = L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^n)$ are variable Lebesgue spaces [40].

Definition 1 [11, 13] The weighted Hardy operators $H_{v(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{v(\cdot)}^{\beta(\cdot)}$ with power weights acting on ψ are defined as

$$H_{v(\cdot)}^{\beta(\cdot)} \psi(t) = t^{\beta(t)+v(t)-1} \int_0^t \frac{\psi(y)}{y^{v(y)}} dy,$$

and

$$\mathcal{H}_{v(\cdot)}^{\beta(\cdot)} \psi(t) = t^{\beta(t)+v(t)} \int_t^\infty \frac{\psi(y)}{y^{v(y)+1}} dy,$$

where $\beta(t)$ and $v(t)$ are measurable functions on $(0, \infty)$.

In the following lemma, one can find minimal conditions on $r^{\frac{\lambda}{\bar{q}(r)}}$ in order that “nice” functions are contained in so-defined spaces.

Lemma 3 [18] Let $1 < p_- \leq p_+ < \infty$, $p \in \mathcal{P}_0(0, \infty)$ and $\lambda \in [0, 1)$. If

$$\sup_{t>0} [\min\{1, t\}]^{\frac{1}{p(0)}} t^{-\frac{\lambda}{\bar{p}(t)}} < \infty, \quad (2.7)$$

then any bounded function f with compact support belongs to $LM_{p(\cdot),\lambda}(0, \infty)$.

Lemma 4 [18] Let $1 < p_- \leq p_+ < \infty$, $p \in \mathcal{P}_{0,\infty}(0, \infty)$, $\lambda \in [0, 1)$, $\lim_{t \rightarrow \infty} t^{\beta(t)}$ exist and is finite and also satisfies (2.7). Assume that the following holds:

- $t^{\beta(t)+v(t)-a}$ and $t^{\frac{\lambda}{p(t)}-v(t)-a}$ are almost decreasing for some $a \in \mathbb{R}$ in case of $H_{v(\cdot)}^{\beta(\cdot)}$.
 - $t^{-\beta(t)-v(t)+b}$ and $t^{\frac{\lambda}{p(t)}-v(t)+b}$ are almost increasing for some $b \in \mathbb{R}$ in case of $\mathcal{H}_{v(\cdot)}^{\beta(\cdot)}$.
- Then

$$v(t) < \frac{\lambda}{p(t)} + \frac{1}{p'(0)}, \quad v(t) > \frac{\lambda}{p(t)} - \frac{1}{p(\infty)}$$

are sufficient for $H_{v(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{v(\cdot)}^{\beta(\cdot)}$, to be defined on the space $LM_{p(\cdot),\lambda}(0, \infty)$, respectively.

The following Hardy inequalities are required for the proofs of our main theorems.

Proposition 1 [18] Let $1 < p_- \leq p_+ < \infty$, $p(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $\lambda \in [0, 1)$. Suppose that (2.7) and Lemma 4 are satisfied. Then $H_{v(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{v(\cdot)}^{\beta(\cdot)}$ are bounded in the space $LM_{p(\cdot),\lambda}(0, \infty)$ if $v(t) < \frac{\lambda}{p(t)} + \frac{1}{p'(t)}$, $v(t) > \frac{\lambda}{p(t)} - \frac{1}{p(t)}$, respectively.

Proposition 2 [18] Let $1 < p_- \leq p_+ < \infty$, $p(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $\lambda \in [0, 1)$. Suppose that (2.7) and Lemma 3 are satisfied. Then $H_{v(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{v(\cdot)}^{\beta(\cdot)}$ are bounded from $LM_{p(\cdot),\lambda}(0, \infty)$ to $LM_{q(\cdot),\lambda}(0, \infty)$ if $v(t) < \frac{\lambda}{p(t)} + \frac{1}{p'(t)}$, $v(t) > \frac{\lambda}{p(t)} - \frac{1}{p(t)}$, respectively.

3 Boundedness of the B -maximal operator in B -local variable Morrey-Lorentz spaces

In this section, we study the boundedness of B -maximal operator in B -local variable Morrey-Lorentz spaces. Firstly, we recall the sharp rearrangement inequality for the B -maximal operator.

Lemma 5 [39] Sharp rearrangement inequality is given by

$$(M_\gamma f)_\gamma^*(t) \leq A f_\gamma^{**}(t), \quad t > 0, \quad (3.1)$$

for B -maximal operator, where $A = A(n, \gamma) > 0$ is a constant.

Theorem 1 Let $p(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, $\lambda \in [0, 1)$ and $f \in \mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. Assume that the assumptions of Lemma 4 and (2.7) are satisfied. Then M_γ is bounded in $\mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.

Proof Let $p(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, $\lambda \in [0, 1)$, $f \in \mathcal{M}_{p(\cdot),q(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ and satisfies (2.7). From the definitions of space

$\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$, average function of f_{γ}^* and (3.1), we obtain

$$\begin{aligned} \|M_{\gamma} f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} (M_{\gamma} f)_{\gamma}^*(t) \right\|_{L_{q(\cdot)}(0, \rho)} \\ &\leq A \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_{\gamma}^{**}(t) \right\|_{L_{q(\cdot)}(0, \rho)} \\ &= A \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)} - 1} \int_0^t f_{\gamma}^*(\tau) d\tau \right\|_{L_{q(\cdot)}(0, \rho)}. \end{aligned}$$

Let $\beta(t) = 0$, $v(t) = \frac{1}{p(t)} - \frac{1}{q(t)}$ and $\psi(t) = t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_{\gamma}^*(t)$. Then, by using the Hardy inequality in Proposition 1 and the definition of space $LM_{q(\cdot), \lambda}(0, \infty)$, we have

$$\begin{aligned} \|M_{\gamma} f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}} &\leq A \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{v(t)} \int_0^t \frac{\psi(\tau)}{\tau^{v(t)}} d\tau \right\|_{L_{q(\cdot)}(0, \rho)} \\ &= A \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| H_{v(t)}^{\beta(t)} \psi \right\|_{L_{q(\cdot)}(0, \rho)} \\ &= A \left\| H_{v(t)}^{\beta(t)} \psi \right\|_{LM_{q(\cdot), \lambda}(0, \infty)} \\ &\leq A \|\psi\|_{LM_{q(\cdot), \lambda}(0, \infty)} \\ &= A \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \|\psi\|_{L_{q(\cdot)}(0, \rho)} \\ &= A \sup_{\rho>0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_{\gamma}^*(t) \right\|_{L_{q(\cdot)}(0, \rho)} \\ &= A \|f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}}. \end{aligned}$$

Hence, the desired result is obtained. $\square$

4 Boundedness of the B -singular integral operator in B -local variable Morrey-Lorentz spaces

This section is devoted the boundedness of B -singular integral operator in B -local variable Morrey-Lorentz spaces. Firstly, we will give two theorems which state O'Neil type inequality for B -convolution operator and a pointwise rearrangement estimate of generalized B -potential integral, respectively.

Theorem 2 [41] *Let f, g be positive measurable functions defined on $\mathbb{R}_{k,+}^n$. Then,*

$$(f \otimes g)_{\gamma}^{**}(t) \leq A_{\gamma, k} \left(f_{\gamma}^{**}(t) \int_0^t g_{\gamma}^{**}(u) du + \int_t^{\infty} f_{\gamma}^*(u) g_{\gamma}^{**}(u) du \right),$$

holds for all $t > 0$.

Theorem 3 [41] *If $K_\alpha \in WL_{\mathcal{Q}/(\mathcal{Q}-\alpha), \gamma}(\mathbb{R}_{k,+}^n)$, $0 < \alpha < \mathcal{Q}$, then*

$$(K_\alpha \otimes g)_\gamma^*(t) \leq (K_\alpha \otimes g)_\gamma^{**}(t) \leq A \left(t^{\alpha/\mathcal{Q}-1} \int_0^t f_\gamma^*(\tau) d\tau + \int_t^\infty \tau^{\alpha/\mathcal{Q}-1} f_\gamma^*(\tau) d\tau \right),$$

where $A = A_{k,\gamma}(\mathcal{Q}/\alpha)^2 \|K_\alpha\|_{WL_{\mathcal{Q}/(\mathcal{Q}-\alpha), \gamma}(\mathbb{R}_{k,+}^n)}$.

Since $T_\gamma f(x)$ is of convolution type and by Theorem 3, one can write

$$(T_\gamma f(x))_\gamma^*(t) \leq (T_\gamma f(x))_\gamma^{**}(t) \leq A \left(t^{-1} \int_0^t f_\gamma^*(\tau) d\tau + \int_t^\infty \tau^{-1} f_\gamma^*(\tau) d\tau \right). \quad (4.1)$$

We are ready to show the $\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ -boundedness of T_γ .

Theorem 4 *Let $p(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $1 < p_- \leq p_+ < \frac{\bar{q}(t)}{\lambda}$, $1 \leq q_- \leq q_+ < \infty$, $\lambda \in [0, 1)$ and $f \in \mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. Assume that the assumptions of Lemma 4 and (2.7) are satisfied. Then T_γ is bounded in $\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*

Proof Let $p(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $1 < p_- \leq p_+ < \frac{\bar{q}(t)}{\lambda}$, $1 \leq q_- \leq q_+ < \infty$, $\lambda \in [0, 1)$ and $f \in \mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. We begin with the definition of the norm in the space $\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$, which is given by

$$\|T_\gamma f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)} = \sup_{\rho > 0} \rho^{-\frac{\lambda}{\bar{q}(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} (T_\gamma f)_\gamma^*(t) \right\|_{L_{q(\cdot)}(0, \rho)}.$$

By substituting the estimate (4.1) into the above norm, we obtain

$$\begin{aligned} & \|T_\gamma f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)} \\ & \leq A \sup_{\rho > 0} \rho^{-\frac{\lambda}{\bar{q}(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \left(t^{-1} \int_0^t f_\gamma^*(\tau) d\tau + \int_t^\infty \tau^{-1} f_\gamma^*(\tau) d\tau \right) \right\|_{L_{q(\cdot)}(0, \rho)} \\ & \leq A \sup_{\rho > 0} \rho^{-\frac{\lambda}{\bar{q}(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)} - 1} \int_0^t f_\gamma^*(\tau) d\tau \right\|_{L_{q(\cdot)}(0, \rho)} \\ & \quad + A \sup_{\rho > 0} \rho^{-\frac{\lambda}{\bar{q}(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \int_t^\infty \tau^{-1} f_\gamma^*(\tau) d\tau \right\|_{L_{q(\cdot)}(0, \rho)} \\ & = I_1 + I_2. \end{aligned}$$

I_1 can easily be obtained with the use of same method as in the proof of Theorem 1, i.e.,

$$I_1 \leq A \|f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)}.$$

For I_2 , choosing $\beta(t) = 0$, $v(t) = \frac{1}{p(t)} - \frac{1}{q(t)}$ and $\psi(t) = t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_{\gamma}^*(t)$ in Proposition 1, we get

$$\begin{aligned}
 I_2 &= \sup_{\rho > 0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \int_t^\infty \tau^{-1} f_{\gamma}^*(\tau) d\tau \right\|_{L_{q(\cdot)}(0, \rho)} \\
 &= \sup_{\rho > 0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{v(t)} \int_t^\infty \frac{\psi(\tau)}{\tau^{v(\tau)+1}} d\tau \right\|_{L_{q(\cdot)}(0, \rho)} \\
 &= A \sup_{\rho > 0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| \mathcal{H}_{v(\cdot)}^{\beta(\cdot)} \psi \right\|_{L_{q(\cdot)}(0, \rho)} \\
 &= A \left\| \mathcal{H}_{v(\cdot)}^{\beta(\cdot)} \psi \right\|_{LM_{q(\cdot), \lambda}(0, \infty)} \\
 &\leq A \|\psi\|_{LM_{q(\cdot), \lambda}(0, \infty)} \\
 &= A \sup_{\rho > 0} \rho^{-\frac{\lambda}{q(\rho)}} \|\psi\|_{L_{q(\cdot)}(0, \rho)} \\
 &= A \sup_{\rho > 0} \rho^{-\frac{\lambda}{q(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_{\gamma}^*(t) \right\|_{L_{q(\cdot)}(0, \rho)} = A \|f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)}.
 \end{aligned}$$

Hence, the desired result is obtained. $\square$

5 Boundedness of the B -Riesz potential in B -local variable Morrey-Lorentz spaces

This section is devoted to the $(\mathcal{M}_{p(\cdot), r(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n), \mathcal{M}_{q(\cdot), s(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n))$ -boundedness of B -Riesz potential.

Firstly, we recall the following inequality, for B -Riesz potential,

$$(I_{\gamma}^{\alpha} f)_{\gamma}^*(t) \leq (I_{\gamma}^{\alpha} f)_{\gamma}^{**}(t) \leq A \left(t^{\frac{\alpha}{Q}-1} \int_0^t f_{\gamma}^*(\tau) d\tau + \int_t^\infty \tau^{\frac{\alpha}{Q}-1} f_{\gamma}^*(\tau) d\tau \right), \quad (5.1)$$

where $A = A_{\gamma, k}(Q/\alpha)^2 \omega(n, k, \gamma)^{(Q-\alpha)/Q}$ and $\omega(n, k, \gamma) = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})}$ [42].

Theorem 5 Let $p(\cdot), q(\cdot), r(\cdot), s(\cdot) \in \mathcal{P}_{0, \infty}(0, \infty)$ satisfying (2.2), $\lambda \in [0, 1)$ and $f \in \mathcal{M}_{p(\cdot), r(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. Suppose also that the assumptions of Lemma 4 and (2.7) are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{Q}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{Q}$, the B -Riesz potential I_{γ}^{α} is bounded from $\mathcal{M}_{p(\cdot), r(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ to $\mathcal{M}_{q(\cdot), s(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.

Proof From the definition of $\mathcal{M}_{q(\cdot),s(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ and (5.1), we get

$$\begin{aligned}
 & \|I_{\gamma}^{\alpha} f\|_{\mathcal{M}_{q(\cdot),s(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)} \\
 &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} (I_{\gamma}^{\alpha} f)_{\gamma}^{*}(t) \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &\leq \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} (I_{\gamma}^{\alpha} f)_{\gamma}^{**}(t) \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &\leq A \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} \left(t^{\frac{\alpha}{Q}-1} \int_0^t f_{\gamma}^{*}(\tau) d\tau + \int_t^{\infty} \tau^{\frac{\alpha}{Q}-1} f_{\gamma}^{*}(\tau) d\tau \right) \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &\leq A \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}+\frac{\alpha}{Q}-1} \int_0^t f_{\gamma}^{*}(\tau) d\tau \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &\quad + A \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} \int_t^{\infty} \tau^{\frac{\alpha}{Q}-1} f_{\gamma}^{*}(\tau) d\tau \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &= J_1 + J_2.
 \end{aligned}$$

By taking into account $v(t) = \frac{1}{p(t)} - \frac{1}{r(t)}$ and $\psi(t) = t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f_{\gamma}^{*}(t)$, we have $\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{Q}$. By Proposition 2, we get $\beta(t) = (1-\lambda) \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right)$, then we find $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right) + \frac{\alpha}{Q}$. Hence $H_{v(\cdot)}^{\beta(\cdot)}$ is bounded from $LM_{r(\cdot),\lambda}(0, \infty)$ to $LM_{s(\cdot),\lambda}(0, \infty)$ under the condition $v(t) = \frac{1}{p(t)} - \frac{1}{r(t)} < \frac{1}{r'(t)} + \frac{\lambda}{\bar{r}(t)}$. From Proposition 2, we obtain

$$\begin{aligned}
 J_1 &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}+\frac{\alpha}{Q}-1} \int_0^t f_{\gamma}^{*}(\tau) d\tau \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\beta(t)+v(t)-1} \int_0^t \frac{\psi(\tau)}{\tau^{v(\tau)}} d\tau \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| H_{v(\cdot)}^{\beta(\cdot)} \psi \right\|_{L_{s(\cdot)}(0,\rho)} \\
 &= \left\| H_{v(\cdot)}^{\beta(\cdot)} \psi \right\|_{LM_{s(\cdot),\lambda}(0,\infty)} \\
 &\leq A \|\psi\|_{LM_{r(\cdot),\lambda}(0,\infty)} \\
 &= A \left\| t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f_{\gamma}^{*}(t) \right\|_{LM_{r(\cdot),\lambda}(0,\infty)} \\
 &= A \sup_{\rho>0} \rho^{-\frac{\lambda}{\bar{r}(\rho)}} \left\| t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f_{\gamma}^{*}(t) \right\|_{L_{r(\cdot)}(0,\rho)} \\
 &= A \|f\|_{\mathcal{M}_{p(\cdot),r(\cdot),\gamma;\lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)}.
 \end{aligned}$$

Now, we estimate J_2 . By taking into account $v(t) = \frac{1}{p(t)} - \frac{1}{r(t)} - \frac{\alpha}{Q}$ and $\psi(t) = t^{\frac{1}{p(t)} - \frac{1}{r(t)}} f_{\gamma}^*(t) \in L_{q(\cdot)}(0, \infty)$, we get $\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{Q}$. By Proposition 2, we find $\beta(t) = (1 - \lambda) \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right)$, then we get $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right) + \frac{\alpha}{Q}$. Hence $\mathcal{H}_{v(\cdot)}^{\beta(\cdot)}$ is bounded from $LM_{r(\cdot), \lambda}(0, \infty)$ to $LM_{s(\cdot), \lambda}(0, \infty)$ under the condition $\frac{\lambda}{r(t)} - \frac{1}{r(t)} < v(t) = \frac{1}{p(t)} - \frac{1}{r(t)} - \frac{\alpha}{Q}$. Then, we obtain

$$\begin{aligned} J_2 &= \sup_{\rho > 0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| t^{\frac{1}{q(t)} - \frac{1}{s(t)}} \int_t^\infty \tau^{\frac{\alpha}{Q} - 1} f_{\gamma}^*(\tau) d\tau \right\|_{L_{s(\cdot)}(0, \rho)} \\ &= \sup_{\rho > 0} \rho^{-\frac{\lambda}{s(\rho)}} \left\| \mathcal{H}_{v(\cdot)}^{\beta(\cdot)} \psi \right\|_{L_{s(\cdot)}(0, \rho)} \\ &= \left\| \mathcal{H}_{v(\cdot)}^{\beta(\cdot)} \psi \right\|_{LM_{s(\cdot), \lambda}(0, \infty)} \\ &\leq A \|\psi\|_{LM_{r(\cdot), \lambda}(0, \infty)} \\ &= A \sup_{\rho > 0} \rho^{-\frac{\lambda}{r(\rho)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{r(t)}} f_{\gamma}^*(t) \right\|_{L_{r(\cdot)}(0, \rho)} \\ &= A \|f\|_{\mathcal{M}_{p(\cdot), r(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k, +}^n)}. \end{aligned}$$

This completes the proof. $\square$

6 Some Applications

Theorems 4 and 5 can be applied to various operators which are estimated from above by T_{γ} and I_{γ}^{α} . Here, we apply these theorems to the Riesz-Bessel transform of high order and B -fractional maximal operator.

A special case of B -singular integral operators is Riesz-Bessel transform of high order. First, let us recall its definition:

$$\begin{aligned} R_{\gamma}^{(k)} f(x) &= A_{\gamma, k} \left[p.v \left(\frac{P_k(y)}{|y|^{k+Q}} \otimes f \right) \right] (x) \\ &= p.v. A_{\gamma, k} \int_{\mathbb{R}_{k, +}^n} \frac{P_k(y)}{|y|^{k+Q}} [T^y f(x)] (y')^v dy \end{aligned}$$

where $A_{\gamma, k} = 2^{Q/2} \frac{\Gamma(Q/2)}{\Gamma(k/2)}$, $k = 1, 2, \dots, n$. Note that

$P_k(y) = P_k(y_1^2, \dots, y_k^2, y_{k+1}, \dots, y_n)$ is a homogeneous polynomial of order k ,

$\Delta_B P_k = 0$ and satisfying the following conditions:

$$\int_{S_{k,+}} P_k(\xi)(\xi')^\gamma d\xi = 0 \quad \text{and} \quad \sup_{\xi \in S_+} |P_k(\xi)| = M < \infty. \quad (6.1)$$

Here the characteristic $P_k(\xi) \in S_{k,+}$, and $\xi = \frac{y}{|y|}$. We denote by $K(y) = \frac{\Theta_k(y)}{|y|^\mathcal{Q}}$,

$\Theta_k(y) = A_{\gamma,k} \frac{P_k(y)}{|y|^k}$ the kernel of Riesz-Bessel transforms of high order [43, 44].

Now, we can give the following result as consequence of our main theorem.

Corollary 1 *Let $1 < p_- \leq p_+ < \frac{\bar{q}(t)}{\lambda}$, $1 \leq q_- \leq q_+ < \infty$ $p(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$, $\lambda \in [0, 1)$ and $f \in \mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. Assume that the assumptions of Lemma 4 and (2.7) are satisfied. Then $R_\gamma^{(k)}$ is bounded in $\mathcal{M}_{p(\cdot), q(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*

The B -fractional maximal operator is defined as

$$M_\gamma^\alpha f(x) = \sup_{r>0} |B_+(0, r)|_\gamma^{\frac{\alpha}{\mathcal{Q}}-1} \int_{B_+(0, r)} T^\gamma |f(x)|(y')^\gamma dy, \quad 0 \leq \alpha < \mathcal{Q}.$$

One can easily observe that $M_\gamma^0 f = M_\gamma f$ for $\alpha = 0$ [45]. Also, the B -fractional maximal operator is closely associated with the B -Riesz potential such that

$$M_\gamma^\alpha \leq A I_\gamma^\alpha. \quad (6.2)$$

Therefore, the next corollary follows immediately.

Corollary 2 *Let $p(\cdot), q(\cdot), r(\cdot), s(\cdot) \in \mathcal{P}_{0,\infty}(0, \infty)$ satisfying (2.2), $\lambda \in [0, 1)$ and $f \in \mathcal{M}_{p(\cdot), r(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. Suppose also that the assumptions of Lemma 4 and (2.7) are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{\mathcal{Q}}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{\mathcal{Q}}$, then M_γ^α is bounded from $\mathcal{M}_{p(\cdot), r(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ to $\mathcal{M}_{q(\cdot), s(\cdot), \gamma; \lambda}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*

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Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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