DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2025-0008
Math. Slovaca **75** (2025), No. 1, 99–112

ON NONOSCILLATION OF FRACTIONAL ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FORCING TERM AND DISTRIBUTED DELAYS

James Viji* — Jehad Alzabut**,c — Velu Muthulakshmi*** — Abdulah Özbekler****

(*Communicated by Jozef Džurina*)

ABSTRACT. The paper deals with the nonoscillatory solutions of arbitrary noninteger-order neutral equations with distributed delays. Through the use of the LFD (Liouville Fractional Derivative) of order $\alpha \geq 0$ on the half-axis and BCP (Banach Contraction Principle), we are able to get the nonoscillation criteria. The obtained results are emphasized with some appropriate examples.

©2024
Mathematical Institute
Slovak Academy of Sciences

1. Introduction

Consider the following fractional neutral equations with distributed delays

$$\mathcal{D}_t^\alpha \Big[\Psi\Big(r(t)y(t) + \int_a^b q(t,\theta)y(t-\theta)\mathrm{d}\theta\Big)\Big]' + \int_c^d p_1(t,\xi)\mathcal{F}_1(y(t-\xi))\mathrm{d}\xi - \int_e^f p_2(t,\mu)\mathcal{F}_2(y(t-\mu))\mathrm{d}\mu = h(t) \tag{1.1}$$

for $t \geq t_0$, where $b > a > 0$, $d > c > 0$, $f > e > 0$ and $\mathcal{D}_t^\alpha$ is the LFD of order $\alpha \geq 0$ on the half-axis.

Further we assume that (1.1) satisfies the conditions:

(1) $r \in \mathrm{C}([t_0,\infty),\mathbb{R}_+)$, $q \in \mathrm{C}([t_0,\infty)\times[a,b],\mathbb{R})$, $h \in \mathrm{C}([t_0,\infty),\mathbb{R})$, $p_1 \in \mathrm{C}([t_0,\infty)\times[c,d],[0,\infty))$, $p_2 \in \mathrm{C}([t_0,\infty)\times[e,f],[0,\infty))$;

(2) $\Psi(u)$ is a continuously increasing real function with respect to u defined on $\mathbb{R}$ and $\Psi^{-1}(u)$ satisfies the local Lipschitz condition with Lipschitz constant K, i.e.,

$$|\Psi^{-1}(u_1) - \Psi^{-1}(u_2)| \leq K\ |u_1 - u_2|.$$

(3) $\mathcal{F}_1\colon \mathbb{R} \to \mathbb{R}$ and $\mathcal{F}_2\colon \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions; $s\mathcal{F}_1(s) > 0$ and $s\mathcal{F}_2(s) > 0$ for $s \neq 0$.

2020 Mathematics Subject Classification: Primary 26A33; Secondary 34C10, 34A08.
Keywords: Nonoscillatory solutions, Liouville fractional derivative, distributed delays, neutral differential equation, Banach contraction principle.
J. Viji was supported by DST-INSPIRE Scheme (No.DST/INSPIRE Fellowship /2018/IF180260), New Delhi, India.
J. Alzabut is thankful to Prince Sultan University and OSTİM Technical University for their endless support.
c Corresponding author.

AUTHOR COPY

The continuous function $y\colon [t_1 - m, \infty) \to \mathbb{R}$ is said to be a solution of equation (1.1) if it satisfies equation (1.1) for $t \geq t_1$, and that

$$\mathcal{D}_t^{\alpha}\Big[\Psi\Big(r(t)y(t) + \int_a^b q(t,\theta)\ y(t-\theta)\mathrm{d}\theta\Big)\Big]'$$

exists on $[t_1, \infty)$, where $m = \max\{b, d, f\}$. A nontrivial solution y of (1.1) is oscillatory if there exist a sequence $\{\nu_n\}$ of real numbers such that $y(\nu_n) = 0$ with $\lim\limits_{n\to\infty} \nu_n = \infty$; otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory provided all solutions of it are oscillatory.

Over last two decades, qualitative behavior of different classes of integer and/or fractional differential equations (FDEs) became very popular, see [1,10,12,13,17,18,21,31,36]. In particular, oscillation theory in FDEs started by Grace et al. [15], and then many researchers have established oscillation results for FDEs; see, for instance, [3,4,13,31,32] and the recent review paper [2].

The literature on neutral delay differential equations has expanded rapidly in recent years. It is a field with fascinating applications to real-world problems. The study of neutral delay differential equations is of great importance from both theoretical and practical perspectives, as they have a wide range of applications across various fields. For instance, neutral delay differential equations are used in modeling lossless transmission line networks, studying vibrating masses on elastic bars, solving variational problems, and neuromechanical systems in which inertia plays a crucial role [5,19,20].

The theory of differential equations is significantly influenced by the concept of nonoscillation. The rationale for this is that the behavior of the solution(s) of an oscillatory equation is entirely obvious; that is, the solutions have infinite zeros. Nevertheless, the equation's nonoscillatory character raises a multitude of additional inquiries, including the sign attitude (positive, negative, ultimately positive, or negative), stability, asymptotic stability, and a variety of other behaviors of the solutions. This results in an increased level of interest in the examination of the solution character.

The existence of nonoscillatory solutions for neutral differential equations with positive and negative coefficients has been the subject of numerous research endeavors thus far. In 2005, Zhang et al. [33] investigated the existence of nonoscillatory solutions of the first-order linear neutral delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) + P(t)x(t-\tau)\Big] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0.$$

In 2012, Candan [7] examined the higher-order nonlinear differential equations of the form

$$\Big[r(t)[x(t) + P(t)x(t-\tau)]^{(n-1)}\Big]' + (-1)^n\Big[Q_1(t)g_1(x(t-\sigma_1)) - Q_2(t)g_2(x(t-\mu)) - f(t)\Big] = 0.$$

In 2013, Candan [8] studied the existence of nonoscillatory solutions for the system of higher-order nonlinear neutral differential equations of the form

$$\Big[\mathbf{x}(t) + P(t)\mathbf{x}(t-\theta)\Big]^{(n)} + (-1)^{n+1}\Big[Q_1(t)\mathbf{x}(t-\sigma_1) - Q_2(t)\mathbf{x}(t-\sigma_2)\Big] = \mathbf{0}.$$

In 2013, Liu et al. [25] have obtained the existence of nonoscillatory solutions for the system of higher-order neutral differential equations of the form

$$\Big[r(t)[\mathbf{x}(t) + P(t)\mathbf{x}(t-\theta)]^{(n-1)}\Big]' + (-1)^n\Big(\int_c^d Q_1(t,\tau)\mathbf{x}(t-\tau)\mathrm{d}\tau - \int_e^f Q_2(t,\sigma)\mathbf{x}(t-\sigma)\mathrm{d}\sigma\Big) = \mathbf{0}.$$

From the literature above, one can observe that the delay term appeared in the neutral part of the most of the differential equations were all constant delays. Due to the challenges in analysis,

AUTHOR COPY

there are few papers addressing the existence of nonoscillatory solutions for differential equations with distributed delays.

In 2013, Candan [9] studied the existence of nonoscillatory solutions of the first-order nonlinear neutral differential equation with distributed deviating arguments of the form

$$\left[\left[x(t)-\int_a^b p_2(t,\xi)\ x(t-\xi)\mathrm{d}\xi\right]^{\gamma}\right]' + \int_c^d Q_2(t,\xi)\ G(x(t-\xi))\mathrm{d}\xi = 0.$$

However, the discussion only covered the condition for the coefficient to be

$$0 < \int_a^b p_2(t,\xi)\mathrm{d}\xi < 1.$$

In 2015, Candan and Gecgel [11] investigated the system of higher-order neutral differential equations with distributed delay of the form

$$\left[x(t)+\int_{a_3}^{b_3}\widetilde{P}(t,\xi)\ x(t-\xi)\mathrm{d}\xi\right]^{(n)} + (-1)^{n+1}\left[\int_{a_1}^{b_1} Q_1(t,\xi)\ x(t-\xi)\mathrm{d}\xi - \int_{a_2}^{b_2} Q_2(t,\xi)\ x(t-\xi)\mathrm{d}\xi\right] = 0,$$

under the conditions

$$0 < \int_{a_3}^{b_3}\widetilde{P}(t,\xi)\mathrm{d}\xi < \frac{1}{2} \qquad \text{and} \qquad -\frac{1}{2} < \int_{a_3}^{b_3}\widetilde{P}(t,\xi)\mathrm{d}\xi < 0.$$

In 2017, Liu et al. [26] established some sufficient conditions for the existence of nonoscillatory solutions of the system of higher-order neutral differential equations with distributed coefficients and delays

$$\left[r(t)\mathbf{x}(t)+\int_a^b p(t,\theta)\ \mathbf{x}(t-\theta)\mathrm{d}\theta\right]^{(n)} + (-1)^{n+1}\Big[Q_1(t)\ \mathbf{x}(t-\tau) - Q_2(t)\ \mathbf{x}(t-\sigma) - h(t)\Big] = \mathbf{0},$$

under the conditions that the coefficient $\int_a^b p(t,\theta)\mathrm{d}\theta$ in the neutral part was discussed in four circumstances, namely, $(-\infty,-1)$, $(-1,0]$, $[0,1)$, $(1,+\infty)$.

Despite this, there have been relatively few papers on nonoscillation of noninteger-order neutral equations, see for instance, references [16, 27, 29, 30, 34, 35]. It is therefore a fascinating area of research to find out if there are nonoscillatory solutions. The idea of this work comes from reading [35], in which the authors looked at the nonoscillation of the fractional order neutral equations of the form

$$\mathcal{D}_t^{\alpha}[x(t)+qx(t-\tau)]' + \sum_{j=1}^{m}\mathcal{Q}_j(t)\mathcal{G}_j(x(t-\sigma_j)) = 0, \tag{1.2}$$

where the neutral coefficient $q \in \mathbb{R} - \{\pm 1\}$.

Compared to discrete delay equations, there is no doubt that FDEs with distributed delay equations are more intricate, precise, and challenging to solve; see for instance [24, 26, 28]. Apart from that, differential equations with forces idealized by nonlocal and/or taxis-driven factors are used to define several science and engineering problems, particularly in mathematical biology and physics, for which integro-differential FDEs are highly helpful in modeling. [6, 14, 22, 23]. To the best of our knowledge, there is limited research on the topic of nonoscillaton criterion of FDEs with distributed delays.

AUTHOR COPY

For the completeness of the paper, we recall some known basic definitions.

Definition 1 (LFI [18]). Let $\alpha \in [0, \infty)$. Then LFI (Liouville Fractional Integral) on the half-axis is defined as

$$\mathcal{D}_t^{-\alpha}\mathcal{G}(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1}\mathcal{G}(s)\mathrm{d}s \qquad \text{for } t \in \mathbb{R}.$$

Definition 2 (LFD [18]). Let $\alpha \in (0, \infty)$. Then LFD on the half-axis is defined as

$$\mathcal{D}_t^{\alpha}\mathcal{G}(t) = \frac{1}{\Gamma(n-\alpha)}(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(\int_t^\infty (s-t)^{n-\alpha-1}\mathcal{G}(s)\mathrm{d}s\Big)$$

for $t \in \mathbb{R}$, where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

When $\alpha = n \in \mathbb{N}$, $\mathcal{D}_t^n\mathcal{G}(t) = (-1)^n\mathcal{G}^{(n)}(t)$ and that $\mathcal{D}_t^{\alpha}(\mathcal{D}_t^{-\alpha}\mathcal{G}(t)) = \mathcal{G}(t)$ is satisfied.

Lemma 1.1 (BCP [1]). *A contraction mapping on a complete metric space has exactly one fixed point.*

2. Main results

In this section, we give some criteria related with the nonoscillation of equation (1.1) under various ranges of

$$\int_a^b q(t,\theta)\mathrm{d}\theta.$$

Let the set Φ be defined as

$$\Phi = \{\varphi : \varphi \in \mathrm{C}[t_0, \infty), \|\varphi\| < \infty\}, \qquad \text{where } \|\varphi\| := \sup_{t \geq t_0} |\varphi(t)|.$$

Then, Φ is, in fact, a Banach space.

For our convenience, pick

$$\mathcal{P}_1(s) = \int_c^d p_1(s,\xi)\mathrm{d}\xi \qquad \text{and} \qquad \mathcal{P}_2(s) = \int_e^f p_2(s,\mu)\mathrm{d}\mu.$$

Theorem 2.1. *Let*

$$0 \leq \int_a^b q(t,\theta)\mathrm{d}\theta \leq q_1 < 1$$

and $1 \leq r(t) \leq 1/q_1$. *Assume that*

$$\int_{t_0}^\infty s^{\alpha}h(s)\mathrm{d}s < \infty \qquad \textit{and} \qquad \int_{t_0}^\infty s^{\alpha}\mathcal{P}_i(s)\mathrm{d}s < \infty, \qquad i = 1, 2. \tag{2.1}$$

Then equation (1.1) *has a nonoscillatory solution which is bounded.*

AUTHOR COPY

P r o o f. Due to (2.1), we find a $t_1 \geq t_0 + m$ large enough such that

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha \Big[\gamma_1\ \mathcal{P}_1(s) + |h(s)|\Big]\mathrm{d}s \leq \Psi(\mathcal{C}_2) - \Psi(\eta), \qquad t \geq t_1; \tag{2.2}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha \Big[\gamma_2\ \mathcal{P}_2(s) + |h(s)|\Big]\mathrm{d}s \leq \Psi(\eta) - \Psi\Big(q_1\mathcal{C}_2 + \frac{\mathcal{C}_1}{q_1}\Big), \qquad t \geq t_1; \tag{2.3}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha \Big(\mathcal{P}_1(s) + \mathcal{P}_2(s)\Big)\mathrm{d}s < \frac{1-q_1}{2KL}, \qquad t \geq t_1, \tag{2.4}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$ are two positive constants such that $q_1\mathcal{C}_2 + \frac{\mathcal{C}_1}{q_1} < \eta < \mathcal{C}_2$, $L = \max\{L_1, L_2\}$; L_1 and L_2 are the Lipschitz constants for the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ on $[\mathcal{C}_1, \mathcal{C}_2]$, respectively, and

$$\gamma_i = \max_{y\in[\mathcal{C}_1,\mathcal{C}_2]}\{\mathcal{F}_i(y)\}, \qquad i = 1, 2.$$

Now, we define the set

$$B_1 = \{y \in \Phi :\ \mathcal{C}_1 \leq y(t) \leq \mathcal{C}_2,\ t \geq t_1\}$$

which is closed and bounded. Therefore, B_1 is complete. Next, we define an operator $G\colon B_1 \to \Phi$ as

$$(Gy)(t) = \begin{cases} r^{-1}(t)\Big\{ -\displaystyle\int_a^b q(t,\theta)y(t-\theta)\mathrm{d}\theta \\ \qquad + \Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\displaystyle\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\mathcal{F}_1(y(s-\xi))\mathrm{d}\xi \\ \qquad - \displaystyle\int_e^f p_2(s,\mu)\mathcal{F}_2(y(s-\mu))\mathrm{d}\mu - h(s)\Big)\mathrm{d}s\ \Big]\Big\}, \qquad t \geq t_1; \\ (Gy)(t_1), \qquad t_0 \leq t \leq t_1. \end{cases} \tag{2.5}$$

Meanwhile, prominently, equation (1.1) has a nonoscillatory solution provided that G has a fixed point in Φ. Lemma 1.1 needs to be proven in three substeps.

Step 1: *G is continuous.* It is not difficult to show that Gy is continuous.

Step 2: $G(B_1) \subset B_1$. For every $y \in B_1$ and using (2.2) for $t \geq t_1$, we have

$$\begin{aligned}(Gy)(t) &\leq r^{-1}(t)\Big\{\Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\mathcal{F}_1(y(s-\xi))\mathrm{d}\xi - h(s)\Big)\mathrm{d}s\Big]\Big\} \\ &\leq r^{-1}(t)\Big\{\Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\int_c^d p_1(s,\xi)\mathrm{d}\xi + |h(s)|\Big)\mathrm{d}s\Big]\Big\} \\ &\leq \Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\ \mathcal{P}_1(s) + |h(s)|\Big)\mathrm{d}s\Big] \\ &\leq \mathcal{C}_2.\end{aligned}$$

AUTHOR COPY

Further, using (2.3) in (2.5) for $t \geq t_1$, we obtain

$$(Gy)(t) \geq r^{-1}(t)\Big\{-\int_a^b q(t,\theta)y(t-\theta)\mathrm{d}\theta + \Psi^{-1}\Big[\Psi(\eta)+\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(-\int_e^f p_2(s,\mu)\mathcal{F}_2(y(s-\mu))\mathrm{d}\mu - h(s)\Big)\mathrm{d}s\Big]\Big\},$$

which implies that

$$\begin{aligned}(Gy)(t) &\geq r^{-1}(t)\Big\{ -q_1\mathcal{C}_2 + \Psi^{-1}\Big[\Psi(\eta)-\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\int_e^f p_2(s,\mu)\mathrm{d}\mu + |h(s)|\Big)\mathrm{d}s\Big]\Big\}\\ &\geq q_1\Big\{ -q_1\mathcal{C}_2 + \Psi^{-1}\Big[\Psi(\eta)-\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\ \mathcal{P}_2(s) + |h(s)|\Big)\mathrm{d}s\Big]\Big\}\\ &\geq \mathcal{C}_1.\end{aligned}$$

Thus, $G(B_1) \subset B_1$.

Step 3: *G is a contraction mapping on B_1.* Let $y_1, y_2 \in B_1$. Then we have

$$\begin{aligned}&\Big|(Gy_1)(t)-(Gy_2)(t)\Big|\\ &\leq r^{-1}(t)\int_a^b q(t,\theta)\Big|y_1(t-\theta)-y_2(t-\theta)\Big|\mathrm{d}\theta\\ &\quad + Kr^{-1}(t)\Big|\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\Big[\int_c^d p_1(s,\xi)\mathcal{F}_1(y_1(s-\xi))\mathrm{d}\xi - \int_e^f p_2(s,\mu)\mathcal{F}_2(y_1(s-\mu))\mathrm{d}\mu\Big]\\ &\quad - \Big[\int_c^d p_1(s,\xi)\mathcal{F}_1(y_2(s-\xi))\mathrm{d}\xi - \int_e^f p_2(s,\mu)\mathcal{F}_2(y_2(s-\mu))\mathrm{d}\mu\Big]\Big)\mathrm{d}s\Big|\end{aligned}$$

for $t \geq t_1$, i.e.,

$$\begin{aligned}&\Big|(Gy_1)(t)-(Gy_2)(t)\Big|\\ &\leq r^{-1}(t)\int_a^b q(t,\theta)\Big|y_1(t-\theta)-y_2(t-\theta)\Big|\mathrm{d}\theta\\ &\quad + \frac{K}{r(t)}\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\Big|\mathcal{F}_1(y_1(s-\xi))-\mathcal{F}_1(y_2(s-\xi))\Big|\mathrm{d}\xi\\ &\quad + \int_e^f p_2(s,\mu)\Big|\mathcal{F}_2(y_1(s-\mu))-\mathcal{F}_2(y_2(s-\mu))\Big|\mathrm{d}\mu\Big)\mathrm{d}s\end{aligned}$$

AUTHOR COPY

$$\le r^{-1}(t)\int_a^b q(t,\theta)\Big|y_1(t-\theta)-y_2(t-\theta)\Big|\mathrm{d}\theta$$
$$+KLr^{-1}(t)\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\Big|y_1(s-\xi)-y_2(s-\xi)\Big|\mathrm{d}\xi$$
$$+\int_e^f p_2(s,\mu)\ \Big|y_1(s-\mu)-y_2(s-\mu)\Big|\mathrm{d}\mu\Big)\mathrm{d}s.$$

Upon using (2.4) in the preceding inequality, we get

$$\Big|(Gy_1)(t)-(Gy_2)(t)\Big| \le \|y_1-y_2\|\left\{q_1+\frac{KL}{\Gamma(\alpha+1)}\int_t^\infty (s-t)^\alpha\Big[\mathcal{P}_1(s)+\mathcal{P}_2(s)\Big]\mathrm{d}s\right\}$$
$$\le \left(\frac{1+q_1}{2}\right)\|y_1-y_2\|.$$

Thus,

$$\|Gy_1-Gy_2\| \le \mathcal{C}\|y_1-y_2\| \qquad \text{for every } y_1,y_2\in B_1.$$

Since $\mathcal{C}=(1+q_1)/2<1$, then G is a contraction on B_1. Thus, $G(\widetilde{y})=\widetilde{y}$, which is indeed a nonoscillatory solution of (1.1). Further, the boundedness of $\widetilde{y}$ comes from the definition of G, because the range of G is Φ. □

Theorem 2.2. *Let* $2q_3\le r(t)\le 2q_2$,

$$1<q_3\le\int_a^b q(t,\theta)\mathrm{d}\theta\le q_2<2q_3<\infty,$$

and (2.1) *be satisfied. Then equation* (1.1) *has a nonoscillatory solution which is bounded.*

P r o o f. Due to (2.1), we find a $t_1\ge t_0+m$ large enough such that

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\ \mathcal{P}_1(s)+|h(s)|\Big)\mathrm{d}s\le\Psi(2q_3\mathcal{C}_4)-\Psi(\eta),\qquad t\ge t_1;\tag{2.6}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\ \mathcal{P}_2(s)+|h(s)|\Big)\mathrm{d}s\le\Psi(\eta)-\Psi\Big(q_2\mathcal{C}_4+2q_2\mathcal{C}_3\Big),\qquad t\ge t_1;\tag{2.7}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big[\mathcal{P}_1(s)+\mathcal{P}_2(s)\Big]\mathrm{d}s<\frac{2q_3-q_2}{2KL},\qquad t\ge t_1,\tag{2.8}$$

where $\mathcal{C}_3$ and $\mathcal{C}_4$ are two positive constants such that

$$q_2\mathcal{C}_4+2q_2\mathcal{C}_3<\eta<2q_3\mathcal{C}_4,$$

$L=\max\{L_1,L_2\}$; L_1 and L_2 are Lipschitz constants of the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ on $[\mathcal{C}_3,\mathcal{C}_4]$, respectively, and

$$\gamma_i=\max_{y\in[\mathcal{C}_3,\mathcal{C}_4]}\{\mathcal{F}_i(y)\},\qquad i=1,2.$$

Now, define the set

$$B_2=\{y\in\Phi:\ \mathcal{C}_3\le y(t)\le\mathcal{C}_4,\ t\ge t_1\}$$

AUTHOR COPY

which is closed and bounded. Therefore, B_2 is complete. Next, we define the operator $G\colon B_2 \to \Phi$ as in (2.5). The remaining part of the proof is similar to that of Theorem 2.1, therefore it is omitted. Thus the proof of Theorem 2.2 is complete. □

Theorem 2.3. *Let* $1 \le r(t) \le -1/q_4$,

$$-1 < q_4 \le \int_a^b q(t,\theta)\mathrm{d}\theta \le 0,$$

and (2.1) *be satisfied. Then equation* (1.1) *has a nonoscillatory solution which is bounded.*

Proof. Due to (2.1), we find a $t_1 \ge t_0 + m$ large enough satisfying

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\mathcal{P}_1(s) + |h(s)|\Big)\mathrm{d}s \le \Psi\Big((1+q_4)\mathcal{C}_6\Big) - \Psi(\eta), \qquad t \ge t_1; \tag{2.9}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\ \mathcal{P}_2(s) + |h(s)|\Big)\mathrm{d}s \le \Psi(\eta) - \Psi\Big(-\frac{\mathcal{C}_5}{q_4}\Big), \qquad t \ge t_1; \tag{2.10}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big[\mathcal{P}_1(s) + \mathcal{P}_2(s)\Big]\mathrm{d}s < \frac{1+q_4}{2KL}, \qquad t \ge t_1, \tag{2.11}$$

where $\mathcal{C}_5$ and $\mathcal{C}_6$ are two positive constants such that

$$-\frac{\mathcal{C}_5}{q_4} < \eta < (1+q_4)\mathcal{C}_6,$$

$L = \max\{L_1, L_2\}$; L_1 and L_i are Lipschitz constants of the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ on $[\mathcal{C}_5, \mathcal{C}_6]$, respectively, and

$$\gamma_i = \max_{y\in[\mathcal{C}_5,\mathcal{C}_6]}\{\mathcal{F}_i(y)\}, \qquad i = 1, 2.$$

Now, define the set

$$B_3 = \{y \in \Phi :\ \mathcal{C}_5 \le y(t) \le \mathcal{C}_6,\ t \ge t_1\}$$

which is closed and bounded. Therefore, B_3 is complete.

Next, we define the operator $G\colon B_3 \to \Phi$ as in (2.5). As a first step in applying Lemma 1.1, we need to separate our proofs into three components:

Step 1: G is continuous. It is not difficult to show that Gy is continuous.

Step 2: $G(B_3) \subset B_3$. Let $y \in B_3$. Using (2.9) we deduce that

$$\begin{aligned}(Gy)(t) &\le r^{-1}(t)\Big\{-\int_a^b q(t,\theta)y(t-\theta)\mathrm{d}\theta \\ &\quad + \Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\mathcal{F}_1(y(s-\xi))d\xi - h(s)\Big)\mathrm{d}s\Big]\Big\} \\ &\le r^{-1}(t)\Big\{-q_4\mathcal{C}_6 + \Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\int_c^d p_1(s,\xi)\mathrm{d}\xi + |h(s)|\Big)\mathrm{d}s\Big]\Big\}\end{aligned}$$

AUTHOR COPY

$$\leq -q_4\mathcal{C}_6 + \Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\ \mathcal{P}_1(s) + |h(s)|\Big)\mathrm{d}s\Big]$$
$$\leq \mathcal{C}_6$$

for $t \geq t_1$. Further using (2.10), (2.5) turns

$$(Gy)(t) \geq r^{-1}(t)\Big\{\Psi^{-1}\Big[\Psi(\eta) + \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(-\int_e^f p_2(s,\mu)\mathcal{F}_2(y(s-\mu))\mathrm{d}\mu - h(s)\Big)\mathrm{d}s\Big]\Big\}$$
$$\geq r^{-1}(t)\Big\{\Psi^{-1}\Big[\Psi(\eta) - \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\int_e^f p_2(s,\mu)d\mu + |h(s)|\Big)\mathrm{d}s\Big]\Big\}$$
$$\geq -q_4\Big\{\Psi^{-1}\Big[\Psi(\eta) - \Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\ \mathcal{P}_2(s) + |h(s)|\Big)\mathrm{d}s\Big]\Big\}$$
$$\geq \mathcal{C}_5.$$

Thus, $G(B_3) \subset B_3$.

Step 3: G *is a contraction mapping on* B_3. Let $y_1, y_2 \in B_3$. Then we have

$$\Big|(Gy_1)(t) - (Gy_2)(t)\Big|$$
$$\leq -r^{-1}(t)\int_a^b q(t,\theta)\Big|y_1(t-\theta) - y_2(t-\theta)\Big|\mathrm{d}\theta$$
$$+ \frac{K}{r(t)}\Big|\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\Big[\int_c^d p_1(s,\xi)\mathcal{F}_1(y_1(s-\xi))\mathrm{d}\xi - \int_e^f p_2(s,\mu)\mathcal{F}_2(y_1(s-\mu))\mathrm{d}\mu\Big]$$
$$- \Big[\int_c^d p_1(s,\xi)\mathcal{F}_1(y_2(s-\xi))\mathrm{d}\xi - \int_e^f p_2(s,\mu)\mathcal{F}_2(y_2(s-\mu))\mathrm{d}\mu\Big]\Big)\mathrm{d}s\Big|$$

for $t \geq t_1$, which yields

$$\Big|(Gy_1)(t) - (Gy_2)(t)\Big|$$
$$\leq -r^{-1}(t)\int_a^b q(t,\theta)\Big|y_1(t-\theta) - y_2(t-\theta)\Big|\mathrm{d}\theta$$
$$+ Kr^{-1}(t)\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\Big|\mathcal{F}_1(y_1(s-\xi)) - \mathcal{F}_1(y_2(s-\xi))\Big|\mathrm{d}\xi$$
$$+ \int_e^f p_2(s,\mu)\Big|\mathcal{F}_2(y_1(s-\mu)) - \mathcal{F}_2(y_2(s-\mu))\Big|\mathrm{d}\mu\Big)\mathrm{d}s$$

AUTHOR COPY

$$\leq -r^{-1}(t)\int_a^b q(t,\theta)\Big|y_1(t-\theta)-y_2(t-\theta)\Big|\mathrm{d}\theta$$
$$+KLr^{-1}(t)\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\int_c^d p_1(s,\xi)\ \Big|y_1(s-\xi)-y_2(s-\xi)\Big|\mathrm{d}\xi$$
$$+\int_e^f p_2(s,\mu)\ \Big|y_1(s-\mu)-y_2(s-\mu)\Big|\mathrm{d}\mu\Big)\mathrm{d}s.$$

Upon using (2.11) in the preceding inequality, we get

$$\Big|(Gy_1)(t)-(Gy_2)(t)\Big| \leq \|y_1-y_2\|\Big\{-q_4+\frac{KL}{\Gamma(\alpha+1)}\int_t^\infty (s-t)^\alpha\Big(\mathcal{P}_1(s)+\mathcal{P}_2(s)\Big)\mathrm{d}s\Big\}$$
$$\leq \left(\frac{1-q_4}{2}\right)\|y_1-y_2\|.$$

Thus,

$$\|Gy_1-Gy_2\| \leq \mathcal{C}\|y_1-y_2\| \qquad \text{for every } y_1,\ y_2\in B_3.$$

Since $\mathcal{C}=(1-q_4)/2<1$, then G is a contraction on B_3. Thus, $G(\bar{y})=\bar{y}$, which is indeed a nonoscillatory solution of (1.1). Further, the boundedness of $\bar{y}$ comes from the definition of G, because the range of G is Φ. □

Theorem 2.4. *Let* $-2q_5 \leq r(t) \leq -2q_6$,

$$-\infty < 2q_5 < q_6 \leq \int_a^b q(t,\theta)\mathrm{d}\theta \leq q_5 < -1,$$

and (2.1) *be satisfied. Then equation* (1.1) *has a nonoscillatory solution which is bounded.*

Proof. Due to (2.1), we find a $t_1 \geq t_0+m$ large enough such that

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_1\ \mathcal{P}_1(s)+|h(s)|\Big)\mathrm{d}s \leq \Psi(-2q_5\mathcal{C}_8+q_6\mathcal{C}_8)-\Psi(\eta), \qquad t\geq t_1; \tag{2.12}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\gamma_2\ \mathcal{P}_2(s)+|h(s)|\Big)\mathrm{d}s \leq \Psi(\eta)-\Psi\Big(-2q_6\mathcal{C}_7\Big), \qquad t\geq t_1; \tag{2.13}$$

$$\Gamma^{-1}(\alpha+1)\int_t^\infty (s-t)^\alpha\Big(\mathcal{P}_1(s)+\mathcal{P}_2(s)\Big)\mathrm{d}s < \frac{q_6-2q_5}{2KL}, \qquad t\geq t_1, \tag{2.14}$$

where $\mathcal{C}_7,\mathcal{C}_8$ are two positive constants such that

$$-2q_6\mathcal{C}_7 < \eta < (-2q_5+q_6)\mathcal{C}_8,$$

$L=\max\{L_1,L_2\}$; L_2 and L_2 are Lipschitz constants of $\mathcal{F}_1$ and $\mathcal{F}_2$ on $[\mathcal{C}_7,\mathcal{C}_8]$, respectively, and

$$\gamma_i = \max_{y\in[\mathcal{C}_7,\mathcal{C}_8]}\{\mathcal{F}_i(y)\}, \qquad i=1,2.$$

Next, define the set

$$B_4=\{y\in\Phi:\ \mathcal{C}_7\leq y(t)\leq \mathcal{C}_8,\ t\geq t_1\}$$

which is closed and bounded. Therefore, B_4 is complete.

AUTHOR COPY

Next, we define the operator $G\colon B_4 \to \Phi$ as in (2.5). The remaining part of the proof is similar to that of Theorem 2.3, therefore it is omitted. Thus the proof of Theorem 2.4 is complete. □

3. Examples

In this section, we give some examples to demonstrate the utility of our main results.

***Example* 1.** Consider

$$\mathcal{D}_t^{1/2}\Big[y(t)+\int\limits_{\pi/2}^{\pi}\mathrm{e}^{-(t+\theta)}y(t-\theta)\mathrm{d}\theta\Big]'+\int\limits_{1}^{2}\sqrt{\xi}\mathrm{e}^{-(\sqrt{t}+\sqrt{\xi})}y^3(t-\xi)\mathrm{d}\xi - \int\limits_{2}^{3}\mathrm{e}^{-(\sqrt{t}+\sqrt{\mu})}y(t-\mu)\mathrm{d}\mu=\mathrm{e}^{-\sqrt{2t}} \tag{3.1}$$

for $t\geq t_0=4$. Here, $\alpha=1/2$, $\Psi(u)=u$, $r(t)=1$, $a=\pi/2$, $b=\pi$, $q(t,\theta)=\mathrm{e}^{-(t+\theta)}$, $c=1$, $d=2$, $\mathcal{F}_1(u)=u^3$, $e=2$, $f=3$, $\mathcal{F}_2(u)=u$, $h(t)=\mathrm{e}^{-\sqrt{2t}}$, and that

$$p_1(t,\xi)=\sqrt{\xi}\mathrm{e}^{-(\sqrt{t}+\sqrt{\xi})} \qquad \text{and} \qquad p_2(t,\mu)=\mathrm{e}^{-(\sqrt{t}+\sqrt{\mu})}.$$

By choosing $q_1=1/4$, one can observe that

$$0\leq\int\limits_{a}^{b}q(t,\theta)\mathrm{d}\theta\leq q_1<1.$$

From (2.1), we get

$$\begin{aligned}
&\int\limits_{t_0}^{\infty}s^{\alpha}\mathcal{P}_1(s)\mathrm{d}s=\int\limits_{4}^{\infty}s^{1/2}\Big(10\mathrm{e}^{-\sqrt{s}-1}-(2^{5/2}+8)\mathrm{e}^{-\sqrt{s}-\sqrt{2}}\Big)\mathrm{d}s<\infty,\\
&\int\limits_{t_0}^{\infty}s^{\alpha}\mathcal{P}_2(s)\mathrm{d}s=\int\limits_{4}^{\infty}s^{1/2}\Big((2^{3/2}+2)\mathrm{e}^{-\sqrt{s}-\sqrt{2}}-(2\sqrt{3}+2)\mathrm{e}^{-\sqrt{s}-\sqrt{3}}\Big)\mathrm{d}s<\infty,\\
&\int\limits_{t_0}^{\infty}s^{\alpha}h(s)\mathrm{d}s=\int\limits_{4}^{\infty}s^{1/2}\mathrm{e}^{-\sqrt{2s}}\mathrm{d}s<\infty.
\end{aligned}$$

Thus, equation (3.1) has a nonoscillatory solution which is bounded via Theorem 2.1.

***Example* 2.** Consider

$$\mathcal{D}_t^{3/2}\left[\frac{1}{2}\Big[\Big(1+\frac{1}{\mathrm{e}^t}\Big)y(t)-\frac{1}{2}\int\limits_{\pi}^{3\pi/2}\mathrm{e}^{-(t+\theta)}\ y(t-\theta)\mathrm{d}\theta\Big]\right]'+\int\limits_{1}^{3}\sqrt{t}\mathrm{e}^{-(t+\xi)}\ y^3(t-\xi)\mathrm{d}\xi - \int\limits_{2}^{4}\frac{\mathrm{e}^{-(t+\mu)}}{\sqrt{t}}\ y^3(t-\mu)\mathrm{d}\mu=\frac{1}{t^3} \tag{3.2}$$

AUTHOR COPY

for $t \geq t_0 = 5$. Here, $\alpha = 3/2$, $\Psi(u) = u/2$, $r(t) = 1 + \mathrm{e}^{-t}$, $a = \pi$, $b = 3\pi/2$, $c = 1$, $d = 3$, $\mathcal{F}_1(u) = \mathcal{F}_2(u) = u^3$, $e = 2$, $f = 4$, $h(t) = t^{-3}$, and that

$$q(t,\theta) = -\frac{1}{2}\mathrm{e}^{-(t+\theta)}, \qquad p_1(t,\xi) = \sqrt{t}\mathrm{e}^{-(t+\xi)} \qquad \text{and} \qquad p_2(t,\mu) = t^{-1/2}\mathrm{e}^{-(t+\mu)}.$$

By choosing $q_4 = -1/2$, one can observe that

$$-1 < q_4 \leq \int_a^b q(t,\theta)\mathrm{d}\theta \leq 0.$$

From (2.1), we get

$$\int_{t_0}^{\infty} s^{\alpha}\mathcal{P}_1(s)\mathrm{d}s = \int_5^{\infty} s^{3/2}(\mathrm{e}^2-1)\sqrt{s}\mathrm{e}^{-s-3}\mathrm{d}s < \infty,$$
$$\int_{t_0}^{\infty} s^{\alpha}\mathcal{P}_2(s)\mathrm{d}s = \int_5^{\infty} s^{3/2}\frac{(\mathrm{e}^2-1)\mathrm{e}^{-s-4}}{\sqrt{s}}\mathrm{d}s < \infty,$$
$$\int_{t_0}^{\infty} s^{\alpha}h(s)\mathrm{d}s = \int_5^{\infty} s^{3/2}\Big(\frac{1}{s^3}\Big)\mathrm{d}s < \infty.$$

Thus, equation (3.2) has a nonoscillatory solution which is bounded via Theorem 2.3.

4. Conclusion

We investigated the existence of nonoscillatory solutions of (1.1) under LFD (Liouville Fractional Derivative) of order $\alpha \geq 0$ on the half-axis. We obtained some sufficient conditions for the existence of nonoscillatory solutions under different ranges of the neutral coefficient $\int_a^b q(t,\theta)\mathrm{d}\theta$ through Banach contraction principle. The obtained results are validated through some examples. Further, under the restrictions given in Examples 1 and 2, the results obtained in the papers [9,11,25,26,28] do not work. This shows that our results are new.

Acknowledgement. The authors sincerely thank the editor and anonymous referees for their careful reading, constructive comments, and valuable suggestions, which have helped to improve the article.

REFERENCES

[1] AGARWAL, R. P.—BOHNER, M.—LI, W. T.: *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, Dekker, New York, 2004.

[2] ALZABUT, J.—AGARWAL, R. P.—GRACE, S. R.—JONNALAGADDA, J. M.—SELVAM, A. G. M.—WANG, C.: *A survey on the oscillation of solutions for fractional difference equations*, Mathematics **10**(6) (2022), Art. No. 894.

[3] ALZABUT, J.—MANIKANDAN, S.—MUTHULAKSHMI, V.—HARIKRISHNAN, S.: *Oscillation criteria for a class of nonlinear conformable fractional damped dynamic equations on time scales*, J. Nonlinear Funct. Anal. **2020** (2020), Art. ID 10.

AUTHOR COPY

[4] ALZABUT, J.—VIJI, J.—MUTHULAKSHMI, V.—SUDSUTAD, W.: *Oscillatory behavior of a type of generalized proportional fractional differential equations with forcing and damping terms*, Mathematics **8**(6) (2020), Art. No. 1037.
[5] BALACHANDRAN, K.—SAKTHIVEL, R.—DAUER, J. P.: *Controllability of neutral functional integrodifferential systems in Banach spaces*, Comput. Math. Appl. **39**(1–2) (2000), 117–126.
[6] BODNAR, M.—PIOTROWSKA, M.: *Stability analysis of the family of tumour angiogenesis models with distributed time delays*, Commun. Nonlinear Sci. Numer. Simul. **31** (2016), 124–142.
[7] CANDAN, T.: *The existence of nonoscillatory solutions of higher order nonlinear neutral differential equations*, Appl. Math. Lett. **25** (2012), 412–416.
[8] CANDAN, T.: *Existence of nonoscillatory solutions for system of higher order neutral differential equations*, Math. Comput. Model. **57**(3–4) (2013), 375–381.
[9] CANDAN T.: *Existence of nonoscillatory solutions of first order nonlinear neutral differential equations*, Appl. Math. Lett. **26** (2013), 1182–1186.
[10] CANDAN, T.: *Existence of nonoscillatory solutions of higher order nonlinear mixed neutral differential equations*, Dyn. Syst. Appl. **27**(4) (2018), 743–755.
[11] CANDAN, T.—GECGEL, A. M.: *Existence of nonoscillatory solutions for system of higher order neutral differential equations with distributed delay*, J. Comput. Anal. Appl. **18**(2) (2015), 266–276.
[12] CHHATRIA, G. N.: *Nonoscillation of first order neutral impulsive difference equations*, Int. J. Difference Equ. **14** (2019), 115–125.
[13] FENG, Q.—MENG, F.: *Oscillation of solutions to nonlinear forced fractional differential equations*, Electron. J. Differential Equations **2013** (2013), Art. No. 169.
[14] FRASSU, S.—VAN DER MEE, C.—VIGLIALORO, G.: *Boundedness in a nonlinear attraction-repulsion Keller-Segel system with production and consumption*, J. Math. Anal. Appl. **504**(2) (2021), Art. ID 125428.
[15] GRACE, S.—AGARWAL, R.—WONG, P. J. Y.—ZAFER, A.: *On the oscillation of fractional differential equations*, Fract. Calc. Appl. Anal. **15**(2) (2012), 222–231.
[16] GRACE, S. R.—GRAEF, J. R.—TUNC, E.: *On the boundedness of nonoscillatory solutions of certain fractional differential equations with positive and negative terms*, Appl. Math. Lett. **97** (2019), 114–120.
[17] GYÖRI, I.—LADAS, G.: *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.
[18] KILBAS, A. A.—SRIVASTAVA, H. M.—TRUJILLO, J. J.: *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[19] KONG, Q.—SUN, Y.—ZHANG, B.: *Nonoscillation of a class of neutral differential equations*, Comput. Math. Appl. **44**(5–6) (2002), 643–654.
[20] KUBIACZYK, I.—SAKER, S. H.—MORCHALO, J.: *New oscillation criteria for first order nonlinear neutral delay differential equations*, Appl. Math. Comput. **142** (2003), 225–242.
[21] LADDE, G. S.—LAKSHMIKANDHAM, V.—ZHANG, B. G.: *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.
[22] LI, T.—PINTUS, N.—VIGLIALORO, G.: *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys. **70**(3) (2019), Art. No. 86.
[23] LI, T.—PINTUS, N.—VIGLIALORO, G.: *Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime*, Differential Integral Equations **34**(5–6) (2021), 315–336.
[24] LI, H.—SUN, S.: *Nonoscillation of higher order mixed differential equations with distributed delays*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), 2617–2625.
[25] LIU, Y.—ZHANG, J.—YAN, Y.: *Existence of nonoscillatory solutions for system of higher-order neutral differential equations with distributed deviating arguments*, Discrete Dyn. Nat. Soc. **2013** (2013), Art. ID 391973.
[26] LIU, Y.—ZHAO, H.—YAN, Y.: *Existence of nonoscillatory solutions for system of higher order neutral differential equations with distributed delays*, Appl. Math. Lett. **67** (2017), 67–74.
[27] LIU, Y.—ZHAO, H.—CHEN, H.—KANG, S.: *Existence of nonoscillatory solutions for system of fractional differential equations with positive and negative coefficients*, J. Appl. Anal. Comput. **9**(5) (2019), 1940–1947.
[28] LIU, Y.—ZHAO, H.—YAN, J.: *Existence of nonoscillatory solutions for system of higher-order neutral differential equations with distributed coefficients and delays*, Adv. Difference Equ. **2017** (2017), Art. No. 41.
[29] MUTHULAKSHMI, V.—PAVITHRA, S.: *Existence of nonoscillatory solutions for mixed neutral fractional differential equation*, Discontinuity Nonlinearity Complex. **9**(1) (2020), 47–61.
[30] MUTHULAKSHMI, V.—PAVITHRA, S.: *Existence of nonoscillatory solutions for fractional neutral functional differential equations*, Malaya J. Mat. **8**(1) (2020), 12–19.
[31] PAVITHRA, S.—MUTHULAKSHMI, V.: *Oscillatory behavior for a class of fractional diffferential equations*, Int. J. Pure Appl. Math. **115**(9) (2017), 93–107.

AUTHOR COPY

[32] SUDSUTAD, W.—ALZABUT, J.—TEARNBUCHA, C.—THAIPRAYOON, C.: *On the oscillation of differential equations in frame of generalized proportional fractional derivatives*, AIMS Math. **5**(2) (2020), 856–871.
[33] ZHANG, W.—FENG, W.—YAN, J.—SONG, J.: *Existence of nonoscillatory solutions of first order linear neutral delay differential equations*, Comput. Math. Appl. **49** (2005), 1021–1027.
[34] ZHOU, Y.—AHMAD, B.—ALSAEDI, A.: *Existence of nonoscillatory solutions for fractional neutral differential equations*, Appl. Math. Lett. **72** (2017), 70–74.
[35] ZHOU, Y.—AHMAD, B.—ALSAEDI, A.: *Existence of nonoscillatory solutions for fractional functional differential equations*, Bull. Malays. Math. Sci. Soc. **42** (2019), 751–766.
[36] ZHOU, Y.—ZHANG, B. G.: *Existence of nonoscillatory solutions of higher order neutral differential equations with positive and negative coefficients*, Appl. Math. Lett. **15** (2002), 867–874.

Received 21. 11. 2023
Accepted 12. 9. 2024

* *Department of Mathematics*
Periyar University
Salem-636 011, Tamilnadu
INDIA
E-mail: vijimaths25@gmail.com

** *Department of Mathematics and Sciences*
Prince Sultan University
11586 Riyadh
SAUDI ARABIA
Department of Industrial Engineering
OSTİM Technical University
06374 Ankara
TÜRKİYE
E-mail: jalzabut@psu.edu.sa; jehad.alzabut@ostimteknik.edu.tr

*** *Department of Mathematics*
Periyar University
Salem-636 011, Tamilnadu
INDIA
E-mail: vmuthumath@periyaruniversity.ac.in

**** *Department of Computer Science*
Faculty of Science
Ankara University
06100 Ankara
TÜRKİYE
E-mail: aozbekler@ankara.edu.tr; aozbekler@gmail.com

AUTHOR COPY