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Research Article

Some properties of the Jost solutions and scattering function of a multiplicative Dirac system

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Abstract

In this study, we deal with a multiplicative Dirac system by giving Jost solutions, Jost functions and multiplicative integral representations of the solutions of this system. We also present asymptotic representations for Jost solutions and Jost functions. Moreover, we define the scattering function of the system.

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1. Introduction

It is known that classical calculus, as developed by Gottfried Leibniz and Isaac Newton in the 17th century, forms the foundation of modern calculus. The fundamental concepts of classical calculus include limits, derivatives, integrals, and series. Since addition and subtraction operations are at the base of these concepts, this calculus is also called additive calculus. Subsequently, with the help of different arithmetic operations, derivative and integral definitions were given and it led alternative analyzes for classical analysis. In 1887, Vito Volterra described the Volterra type analysis [25]. Then, Non-Newtonian analysis was introduced different from classical analysis by Grossman and Katz in 1967 [18,19], and since the beginning of the last century, Non-Newtonian analysis has drawn the interest of numerous researchers. This field is subdivided into various branches, including geometric, anageometric, biogeometric, quadratic, and harmonic analysis, each offering distinct advantages for modeling multiplicative phenomena. One of these subdivisions is geometric analysis, alternatively referred to as multiplicative analysis following Dick Stanley's terminology [24]. Geometric multiplicative calculus is widely used for processes that evolve proportionally, ensuring positivity and naturally describing growth or decay.

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Bigeometric calculus extends this approach to higher-order multiplicative derivatives, allowing more refined modeling of changing growth rates. Newton and Leibniz's approach to analysis relies on addition and subtraction, whereas multiplicative analysis operates on multiplication and division. There are numerous motivations for employing multiplicative calculus. It offers simpler solutions to problems that pose challenges in classical analysis. Moreover, every aspect in classical analysis finds its counterpart in Non-Newtonian analysis. Multiplicative calculus serves as a valuable complement to classical calculus, particularly suited for scenarios involving exponential functions, much like how classical calculus is formed for linear functions. Many events in nature change exponentially, such as the levels of sound signals, the acidity of chemicals and earthquake magnitudes. Hence, employing multiplicative analysis in these contexts proves highly advantageous [8]. Additionally, this analysis yields superior outcomes compared to conventional approaches across diverse fields including finance, economics, biology, demography, image pattern recognition, signal processing, quantum information theory, mathematical modeling, engineering, and medicine [10–13, 22, 27]. Since the beginning of the 2000s, numerous studies have been conducted in multiplicative analysis and important results have been obtained [6, 7, 9, 15, 17, 23, 26, 28, 29]. On the other hand, a Dirac system constitutes an efficient equation system and in classical mathematical physics. The Dirac equation represents a contemporary formulation of relativistic quantum mechanics for electrons, providing valuable and comprehensible mathematical and physical insights. Extensive research has been conducted on this equation system in the classical context, leading to noteworthy findings [1–4, 14, 21]. Furthermore, recent research has explored the multiplicative analysis of the Dirac system [16, 20]. The authors construct the multiplicative Dirac system by arithmetic operations and they make an asymptotic estimates for the eigenfunction of this system in [20]. Moreover, they investigate some spectral properties of this system. The study [16] is conformable fractional form of the study [20]. Unlike the previous studies [16] and [20], this investigation will focus on examining the scattering properties of the Dirac system within the multiplicative case by presenting Jost solutions and Jost functions of the problem, and their asymptotics. This approach is anticipated to yield distinct outcomes compared to the classical case, offering a novel perspective on the subject matter.

The rest of this study is organized as follows:

- Firstly, we give the basic definitions and theorems of multiplicative analysis that we will use and we introduce our problem as a multiplicative Dirac system in Section 2.
- Then, we present multiplicative integral equations as the solutions of the problem. We also show that the system related to the main problem is the solution of these integral equations in Section 3.
- Next, we obtain the Jost solutions of the problem and examine their asymptotic properties in Section 4. We also show in this section that Jost solutions have analytical continuation.
- Finally, we find Jost functions of the problem in Section 5, we define the scattering function of the problem and show its property in this section.

Dirac equations are a set of equations used in quantum mechanics to describe the behavior of the electron, and these equations include the effects of special relativity and take into account the spin of the electron. For this reason, Dirac equations are one of the fundamental building blocks of relativistic quantum mechanics. Dirac equation systems have many applications such as nuclear and high energy physics, particle accelerators, antimatter research, quantum field theories and graphene physics. Numerous studies have explored this system in the classical case. In recent years, due to the growing interest in multiplicative calculus and its applications, we here deal with a multiplicative Dirac system within the structure of multiplicative calculus. The multiplicative Dirac system is significant because

it extends relativistic quantum models to a multiplicative form, enabling a more natural treatment of proportional changes and revealing new analytical properties beyond the classical case.

2. Preliminaries

Before giving the main results, let's give the fundamental concepts and important theorems of the multiplicative analysis that we will use in the study.

Definition 2.1 ([5]). Let $v : \mathcal{A} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable in usual case and $v(\zeta) > 0$ for all ζ. If the below limit exists and positive

$$v^*(\zeta) = \lim_{h \to 0} \left[\frac{v(\zeta + h)}{v(\zeta)} \right]^{\frac{1}{h}},$$

$v^*(\zeta)$ is called multiplicative or $(^*)$ derivative of v at ζ.

Lemma 2.2 ([5]). *Let $v : \mathcal{A} \to \mathbb{R}$ be a positive and usual differentiable at ζ. Then, there is a following relation between classical and * derivatives*

$$v^*(\zeta) = e^{(\ln \circ v)'(\zeta)}.$$

Theorem 2.3 ([5]). *Assume that v, r are * differentiable functions and g is a classical differentiable function at the point ζ. Then following equations are satisfied for * derivative.*

i. $(cv)^*(\zeta) = v^*(\zeta), \quad c \in \mathbb{R}^+,$
ii. $(vr)^*(\zeta) = v^*(\zeta) r^*(\zeta),$
iii. $\left(\frac{v}{r}\right)^*(\zeta) = \frac{v^*(\zeta)}{r^*(\zeta)},$
iv. $(v^g)^*(\zeta) = v^*(\zeta)^{g(\zeta)} v(\zeta)^{g'(\zeta)},$
v. $(vog)^*(\zeta) = v^*(g(\zeta))^{g'(\zeta)},$
vi. $(v+r)^*(\zeta) = v^*(\zeta)^{\frac{v(\zeta)}{v(\zeta)+r(\zeta)}} r^*(\zeta)^{\frac{r(\zeta)}{v(\zeta)+r(\zeta)}}.$

Definition 2.4 ([5]). Let v be a positive, bounded function on $[a, b]$, where $-\infty < a < b < \infty$. Then, the symbol $\int_a^b v(\zeta)^{d\zeta}$ is called multiplicative integral or * integral of v on $[a, b]$. By this definition, if v is positive and Riemann integrable on $[a, b]$, then it is * integrable on $[a, b]$ and can be written

$$\int_a^b v(\zeta)^{d\zeta} = e^{\int_a^b (\ln \circ v)(\zeta) d\zeta}.$$

Conversely, one can show that if v is Riemann integrable on $[a, b]$, then

$$\int_a^b v(\zeta)\, d\zeta = \ln \int_a^b \left(e^{v(\zeta)}\right)^{d\zeta}.$$

Theorem 2.5 ([5]). *Let v, $r > 0$ be * integrable functions on $[a, b]$. Then, below expressions hold:*

(i) $\int_a^b \left[v(\zeta)^k\right]^{d\zeta} = \left[\int_a^b v(\zeta)^{d\zeta}\right]^k,$

(ii) $\int_a^b [v(\zeta) r(\zeta)]^{d\zeta} = \int_a^b v(\zeta)^{d\zeta} \int_a^b r(\zeta)^{d\zeta},$

(iii) $\int_a^b \left[\frac{v(\zeta)}{r(\zeta)}\right]^{d\zeta} = \frac{\int_a^b v(\zeta)^{d\zeta}}{\int_a^b r(\zeta)^{d\zeta}},$

(iv) $\int_a^b v(\zeta)^{d\zeta} = \int_a^c v(\zeta)^{d\zeta} \int_c^b v(\zeta)^{d\zeta},$

where $k \in \mathbb{R}$ *is a constant and* $c \in [a,b]$.

Theorem 2.6 ([5]). *Let* $v : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be multiplicative integrable function and* $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be classical integrable function. Then,* v^u *is multiplicative integrable and it is given by*

$$\int_a^b \left[v^*(\zeta)^{u(\zeta)}\right]^{d\zeta} = \frac{v(b)^{u(b)}}{v(a)^{u(a)}} \left\{\int_a^b \left[v(\zeta)^{u'(\zeta)}\right]^{d\zeta}\right\}^{-1}.$$

Now, we introduce some essential algebraic structures which are necessary for us to establish the main problem. Note that arithmetic operations formed by exponential functions are called multiplicative algebraic (* algebraic) operations. Let us give certain properties of these operations using the * arithmetic table for $v, r \in \mathbb{R}^+$

$$v \oplus r = vr, \quad v \ominus r = \frac{v}{r}, \quad v \odot r = v^{\ln r} = r^{\ln v}.$$

These operations give rise to specific algebraic structures.

Definition 2.7. Let X be a non empty set and $<,>_*: X \times X \to \mathbb{R}^+$ be a function such that below axioms hold for each $t, m, x \in X$

i) $< t, t >_* \geq 1$,
ii) $< t, t >_* = 1$ if $t = 1$,
iii) $< t \oplus m, x >_* = < t, x >_* \oplus < m, x >_*$,
iv) $< e^\alpha \odot t, m >_* = e^\alpha \odot < t, m >_*, \quad \alpha \in \mathbb{R}$,
v) $< t, m >_* = < m, t >_*$.

Then, this mapping is called multiplicative (*) inner product on X and denoted by $<,>_*$. Also the space $(X, <,>_*)$ is called the * inner product space [15].

Note that $L_2^*[a,b] = \left\{ v : \int_a^b [v(\zeta) \odot v(\zeta)]^{d\zeta} < \infty \right\}$ is an * inner product space with multiplicative inner product

$$<,>_*: L_2^*[a,b] \times L_2^*[a,b] \to \mathbb{R}^+, \quad < v, u >_* = \int_a^b [v(\zeta) \odot u(\zeta)]^{d\zeta},$$

where $v, u \in L_2^*[a,b]$ are positive functions [26]. It is clear that this space is the multiplicative analogue of the well-known $L_2[a,b]$.
Furthermore, let $v : [a,b] \to (0,\infty)$ be a positive real-valued function. We say that $v \in L_1^*([a,b])$ if

$$\int_a^b |\ln v(x)|\, dx < \infty.$$

In other words,

$$v \in L_1^*([a,b]) \iff \ln v \in L_1([a,b]).$$

Consider the following multiplicative Dirac system for $\zeta \in [0,\infty)$

$$L(\phi) = [K \odot \phi^*(\zeta)] \oplus [Q(\zeta) \odot \phi(\zeta)] = e^\mu \odot \phi(\zeta) \tag{2.1}$$

with the boundary condition

$$\frac{\phi_1(0,\mu)}{\phi_2(0,\mu)} = 1, \tag{2.2}$$

where

$$K = \begin{bmatrix} e^{i} & 1 \\ 1 & e^{-i} \end{bmatrix}, \; Q(\zeta) = \begin{bmatrix} 1 & e^{q_1(\zeta)} \\ e^{q_2(\zeta)} & 1 \end{bmatrix},$$

μ is a spectral parameter and q_1, q_2 are real-valued, positive and * integrable functions.

$$\phi(\zeta,\mu) = [\phi_1(\zeta,\mu), \phi_2(\zeta,\mu)]^T$$

is the eigenfunction of the problem (2.1)-(2.2) corresponding to the eigenvalue μ, where T denotes transpose of the matrix and ϕ_j, $j = 1, 2$ are continuous and positive functions. The following multiplicative system is obtained by using the properties of multiplicative analysis in (2.1)

$$\phi_1^*(\zeta,\mu)^{i}\, \phi_2^*(\zeta,\mu)^{q_1(\zeta)} = \phi_1(\zeta,\mu)^{\mu} \tag{2.3}$$

$$\phi_2^*(\zeta,\mu)^{-i}\, \phi_1^*(\zeta,\mu)^{q_2(\zeta)} = \phi_2(\zeta,\mu)^{\mu}. \tag{2.4}$$

3. The solutions of multiplicative Dirac system

In this section, we investigate the solutions of the multiplicative Dirac system which are given as * integral equations.

Theorem 3.1. *Each solution of the system of the multiplicative differential equations (2.3)-(2.4) is a solution of the system of the multiplicative integral equations defined by*

$$\phi_1(\zeta,\mu) = \phi_1(0,\mu)^{e^{-i\mu\zeta}} \prod_0^{\zeta} \left(\phi_2(\eta,\mu)^{ie^{-i\mu(\zeta-\eta)q_1(\eta)}}\right)^{d\eta} \tag{3.1}$$

$$\phi_2(\zeta,\mu) = \phi_2(0,\mu)^{e^{i\mu\zeta}} \prod_0^{\zeta} \left(\phi_1(\eta,\mu)^{-ie^{i\mu(\zeta-\eta)q_2(\eta)}}\right)^{d\eta}. \tag{3.2}$$

Furthermore, the opposite of the above expression is also true.

Proof. Firstly, let us show that each solution of the system (2.1)-(2.2) is a solution of (3.1)-(3.2). We use the method of variation of parameters while making the proof. The homogeneous equation of (2.3) is written as

$$\phi_1^*(\zeta,\mu)^{i} = \phi_1(\zeta,\mu)^{\mu}. \tag{3.3}$$

We obtain the general solution of (3.3) as

$$\widetilde{\phi}_1(\zeta,\mu) = e^{c_1 e^{-i\mu\zeta}},$$

where c_1 is a constant. Using the solution $\widetilde{\phi}_1(\zeta,\mu)$, we will look for the general solution of the equation (2.3) as the following form

$$\phi_1(\zeta,\mu) = e^{c_1(\zeta)e^{-i\mu\zeta}}. \tag{3.4}$$

By using (2.3) and (3.4), we get

$$e^{c_1'(\zeta)} = \phi_2(\zeta,\mu)^{iq_1(\zeta)e^{i\mu\zeta}}.$$

By taking the multiplicative integral for both sides of the last equality on $[0,\zeta]$ and using (3.4), it is easy to write

$$\phi_1(\zeta,\mu)=\phi_1(0,\mu)^{e^{-i\mu\zeta}}\int_0^{\zeta}\left(\phi_2(\eta,\mu)^{ie^{-i\mu(\zeta-\eta)q_1(\eta)}}\right)^{d\eta}.$$

Similarly, it can be easily shown that the solution of the equation (2.4) satisfies the multiplicative integral equation (3.2). Now let us prove the reverse. By taking the multiplicative derivative of (3.1) in both sides, we find

$$\phi_1^*(\zeta,\mu)=\left[e^{\ln\phi_1(0,\mu)^{e^{-i\mu\zeta}}\int_0^{\zeta}\left(\phi_2(\eta,\mu)^{ie^{-i\mu(\zeta-\eta)q_1(\eta)}}\right)^{d\eta}}\right]^{-i\mu}\left(e^{\ln\phi_2(\zeta,\mu)^{q_1(\zeta)}}\right)^{i}.$$

From (3.1) and the last equality, the following can be easily found

$$\phi_1^*(\zeta,\mu)^{i}\,\phi_2(\zeta,\mu)^{q_1(\zeta)}=\phi_1(\zeta,\mu)^{\mu}.$$

Similarly to this one, it can be easily obtained that the solution of (3.2) satisfies the equation (2.4). □

4. Jost solutions of multiplicative Dirac system

In this section, firstly, we obtain the Jost solutions of the problem (2.1)-(2.2). Then, we examine the properties of these Jost solutions and find asymptotic equations related to them. Let's make some preparations to get the results that we desire.

$$\tau(\zeta,\mu)=\begin{bmatrix} e^{\widetilde{A}e^{-i\mu\zeta}} \\ e^{\widetilde{B}e^{i\mu\zeta}} \end{bmatrix};\ \ \widetilde{A},\widetilde{B}\in\mathbb{R}$$

is the solution of the multiplicative Dirac system (2.3)-(2.4) for $q_1(\zeta)=q_2(\zeta)=0$. Assume that functions $q_1(\zeta)$ and $q_2(\zeta)$ satisfy the following condition

$$|q_i(\zeta)|\le\frac{C}{(1+\zeta)^{1+\varepsilon}},\ \ i=1,2,$$

where $C>0$ and $\varepsilon>0$. We will show the solution of the system (2.3)-(2.4) that fulfills the following condition

$$\phi(\zeta,\mu)=\begin{bmatrix}\phi_1(\zeta,\mu)\\ \phi_2(\zeta,\mu)\end{bmatrix}=\begin{bmatrix} e^{\widetilde{A}e^{-i\mu\zeta}} \\ e^{\widetilde{B}e^{i\mu\zeta}} \end{bmatrix}+o(1),\ \ \zeta\to\infty \tag{4.1}$$

by $\phi(\zeta,\mu)$.

Theorem 4.1. *The solution of the Dirac multiplicative system (2.3)-(2.4) satisfying the condition (4.1) is also a solution of the integral equations system given by*

$$\phi_1(\zeta,\mu)=e^{\widetilde{A}e^{-i\mu\zeta}}\int_{\zeta}^{\infty}\left(\phi_2(\eta,\mu)^{-ie^{-i\mu(\zeta-\eta)q_1(\eta)}}\right)^{d\eta} \tag{4.2}$$

$$\phi_2(\zeta,\mu)=e^{\widetilde{B}e^{i\mu\zeta}}\int_{\zeta}^{\infty}\left(\phi_1(\eta,\mu)^{ie^{i\mu(\zeta-\eta)q_2(\eta)}}\right)^{d\eta}. \tag{4.3}$$

Proof. Since the general solution of the equation (2.3) is

$$\phi_1(\zeta,\mu)=e^{d_1(\zeta)e^{-i\mu\zeta}}, \tag{4.4}$$

it can be easily found that

$$e^{d_1'(\zeta)}=\phi_2(\zeta,\mu)^{ie^{i\mu\zeta}q_1(\zeta)}.$$

by the help of (2.3) and (4.4), where $d_1(\zeta)$ is a constant. By taking the multiplicative integral for both sides of the last equality from ζ to infinity and using (4.2), we get

$$\phi_1(\zeta,\mu) = e^{\widetilde{A}e^{-i\mu\zeta}} \int_{\zeta}^{\infty} \left(\phi_2(\eta,\mu)^{-ie^{-i\mu(\zeta-\eta)}q_1(\eta)}\right)^{d\eta},$$

here $\widetilde{A} = \lim_{\zeta\to\infty} d_1(\zeta)$ is a real number. It gives the one part of the proof. Another part can be obtained in a similar way. □

Theorem 4.2. *Each solution of the multiplicative integral equations (4.2)-(4.3) is the solution of (2.3)-(2.4) under the condition (4.1).*

Proof. It is easily seen that the equations (4.2) and (4.3) satisfy the condition (4.1). Taking the multiplicative derivative of both sides of (4.2), we write

$$\phi_1^*(\zeta,\mu) = \left[e^{\widetilde{A}e^{-i\mu\zeta}} \int_{\zeta}^{\infty} \left(\phi_2(\eta,\mu)^{-ie^{-i\mu(\zeta-\eta)}q_1(\eta)}\right)^{d\eta}\right]^{-i\mu} \left(e^{\ln\phi_2(\zeta,\mu)q_1(\zeta)}\right)^{i}.$$

By using (4.2) and the last equality, we find

$$\phi_1^*(\zeta,\mu)^{i}\,\phi_2(\zeta,\mu)^{q_1(\zeta)} = \phi_1(\zeta,\mu)^{\mu}.$$

In a similar way, it can shown that, each solution of the multiplicative integral equation (4.3) is the solution of the equation (2.4). □

Note that for $\mu\in\mathbb{R}$, we also write the system of the multiplicative integral equations (4.2)-(4.3) as follows

$$\phi(\zeta,\mu) = \tau(\zeta,\mu) \oplus \int_{\zeta}^{\infty} \left[\mathbf{R}(\zeta,\eta;\mu)\odot\phi(\eta,\mu)\right]^{d\eta}, \tag{4.5}$$

where

$$\mathbf{R}(\zeta,\eta;\mu) = \begin{bmatrix} 1 & e^{-iq_1(\eta)e^{-i\mu(\zeta-\eta)}} \\ e^{iq_2(\eta)e^{i\mu(\zeta-\eta)}} & 1 \end{bmatrix}.$$

The solution of the multiplicative equation has the following integral representation

$$\phi(\zeta,\mu) = \tau(\zeta,\mu) \oplus \int_{\zeta}^{\infty} \left[\Psi(\zeta,s)\odot\tau(s,\mu)\right]^{ds},$$

where $\Psi(\zeta,s)$ is defined by

$$\Psi(\zeta,s) = \begin{bmatrix} \Psi_{11}(\zeta,s) & \Psi_{12}(\zeta,s) \\ \Psi_{21}(\zeta,s) & \Psi_{22}(\zeta,s) \end{bmatrix}$$

and provides the following inequality

$$|\ln\Psi_{ij}(\zeta,s)| \le \frac{k}{(1+\zeta+s)^{1+\varepsilon}}, \quad k>0, \quad \varepsilon>0, \quad i,j=1,2. \tag{4.6}$$

We get the Jost solutions of the main multiplicative Dirac system by using the solution $\phi(\zeta,\mu)$. It is clear from (4.5) that we write the solution $\phi(\zeta,\mu)$ as the following form for $\mu\in\mathbb{R}$

$$\begin{cases} \phi_1(\zeta,\mu) = e^{\widetilde{A}e^{-i\mu\zeta}} \int_{\zeta}^{\infty} \left(\Psi_{11}(\zeta,s)^{\widetilde{A}e^{-i\mu s}}\right)^{ds} \int_{\zeta}^{\infty} \left(\Psi_{12}(\zeta,s)^{\widetilde{B}e^{i\mu s}}\right)^{ds} \\ \phi_2(\zeta,\mu) = e^{\widetilde{B}e^{i\mu\zeta}} \int_{\zeta}^{\infty} \left(\Psi_{21}(\zeta,s)^{\widetilde{A}e^{-i\mu s}}\right)^{ds} \int_{\zeta}^{\infty} \left(\Psi_{22}(\zeta,s)^{\widetilde{B}e^{i\mu s}}\right)^{ds}. \end{cases} \tag{4.7}$$

Now, we examine two cases to get the Jost solutions of (2.1)-(2.2).

Case 1: Let us take $\widetilde{A}=0$ and $\widetilde{B}=1$ in (4.7). Then, we find the Jost solution of (2.1)-(2.2) for $\mu\in\mathbb{R}$ as

$$\phi(\zeta,\mu)=\Phi^{+}(\zeta,\mu)=\begin{bmatrix}\Phi_1^{+}(\zeta,\mu)\\ \Phi_2^{+}(\zeta,\mu)\end{bmatrix}, \tag{4.8}$$

where

$$\Phi_1^{+}(\zeta,\mu)=\int_{\zeta}^{\infty}\left(\Psi_{12}(\zeta,s)^{e^{i\mu s}}\right)^{ds},$$

$$\Phi_2^{+}(\zeta,\mu)=e^{e^{i\mu\zeta}}\int_{\zeta}^{\infty}\left(\Psi_{22}(\zeta,s)^{e^{i\mu s}}\right)^{ds}.$$

Case 2: Let us take $\widetilde{A}=1$ and $\widetilde{B}=0$ in (4.7). Then, we find the Jost solution of the problem (2.1)-(2.2) for $\mu\in\mathbb{R}$ as follows

$$\phi(\zeta,\mu)=\Phi^{-}(\zeta,\mu)=\begin{bmatrix}\Phi_1^{-}(\zeta,\mu)\\ \Phi_2^{-}(\zeta,\mu)\end{bmatrix}, \tag{4.9}$$

where

$$\Phi_1^{-}(\zeta,\mu)=e^{e^{-i\mu\zeta}}\int_{\zeta}^{\infty}\left(\Psi_{11}(\zeta,s)^{e^{-i\mu s}}\right)^{ds},$$

$$\Phi_2^{-}(\zeta,\mu)=\int_{\zeta}^{\infty}\left(\Psi_{21}(\zeta,s)^{e^{-i\mu s}}\right)^{ds}.$$

Theorem 4.3. *For $i,j=1,2$, we find $\Psi_{ij}(\zeta,\cdot)$, $\Psi_{ij}(\cdot,s)\in L_1^{*}(0,\infty)$.*

Proof. By using the definition of the multiplicative integral and (4.6), it is easy for us to see

$$\int_0^{\infty}|\Psi_{ij}(\zeta,s)|^{ds}=e^{\int_0^{\infty}\ln|\Psi_{ij}(\zeta,s)|^{ds}}\leq e^{\int_0^{\infty}\frac{k}{(1+\zeta+s)^{1+\varepsilon}}ds}<\infty.$$

It follows from that $\Psi_{ij}(\zeta,\cdot)\in L_1^{*}(0,\infty)$. The proof of $\Psi_{ij}(\cdot,s)\in L_1^{*}(0,\infty)$ can be obtained similarly, so it completes the proof of theorem. □

Theorem 4.4. *For $\mu\in\mathbb{R}$, the Jost solutions $\Phi^{+}(\zeta,\mu)$ and $\Phi^{-}(\zeta,\mu)$ have analytic continuation to $\mathbb{C}_{+}$ open-upper complex half-plane and $\mathbb{C}_{-}$ open-lower complex half-plane with respect to μ, respectively.*

Proof. For $\mu\in\mathbb{C}_{+}$, by the help of Definition 2.4 and (4.6), we see that

$$\left|\int_{\zeta}^{\infty}\left(\Psi_{12}(\zeta,s)^{e^{i\mu s}}\right)^{ds}\right|\leq e^{\int_{\zeta}^{\infty}\left|e^{i\mu s}\right|\ln|\Psi_{12}(\zeta,s)|ds}$$

$$\leq e^{\int_{\zeta}^{\infty}\left|e^{i\mu s}\right|\frac{c}{(1+\zeta+s)^{1+\varepsilon}}ds}<\infty.$$

It gives that $\Phi_1^{+}(\zeta,\mu)$ converges uniformly. So, we write

$$\frac{d^{*}}{d\mu}\int_{\zeta}^{\infty}\left(\Psi_{12}(\zeta,\mu)^{e^{i\mu s}}\right)^{ds}=e^{\frac{d}{d\mu}\int_{\zeta}^{\infty}\left(e^{i\mu s}\ln\Psi_{12}(\zeta,s)\right)ds}.$$

Using the last equality and (4.6), we find

$$\left| e^{\int_{\zeta}^{\infty} is e^{i\mu s} \ln \Psi_{12}(\zeta,s) ds} \right| < \infty.$$

By using the Definition 2.4 and (4.6), it is obtained that

$$\left| e^{e^{i\mu\zeta} \int_{\zeta}^{\infty} \left(\Psi_{22}(\zeta,s)^{e^{i\mu s}} \right)^{ds}} \right| < \infty.$$

Using the multiplicative derivative, the multiplicative integral and (4.6), it is shown that the derivative of the function $\Phi_2^+(\zeta,\mu)$ is finite. Then, the Jost solution $\Phi^+(\zeta,\mu)$ has analytic continuation from $\mathbb{R}$ to the $\mathbb{C}_+$ with respect to μ. In a similar way, it can be shown that the Jost solution $\Phi^-(\zeta,\mu)$ has analytic continuation from $\mathbb{R}$ to the $\mathbb{C}_-$ with respect to μ. □

As a result of above theorem, we write the solutions (4.8) and (4.9) for $\mu \in \overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. These are called the Jost solutions of the multiplicative Dirac system (4.8)-(4.9).

Theorem 4.5. *The Jost solutions $\Phi^+(\zeta,\mu)$ and $\Phi^-(\zeta,\mu)$ satisfy the following asymptotic representations*

$$\ln \Phi^+(\zeta,\mu) = \begin{bmatrix} 0 \\ e^{i\mu\zeta} \end{bmatrix} + o(1), \quad \mu \in \overline{\mathbb{C}}_+, \quad \zeta \to \infty$$

and

$$\ln \Phi^-(\zeta,\mu) = \begin{bmatrix} e^{-i\mu\zeta} \\ 0 \end{bmatrix} + o(1), \quad \mu \in \overline{\mathbb{C}}_-, \quad \zeta \to \infty.$$

Proof. From Definition 2.4 and (4.6), we find

$$\Phi_1^+(\zeta,\mu) = e^{o(1)}$$
$$\Phi_2^+(\zeta,\mu) = e^{e^{i\mu\zeta}+o(1)}$$

for $\mu \in \overline{\mathbb{C}}_+$ whenever $\zeta \to \infty$. It gives the one part of the proof. In a similar way, the desired asymptotic equation for $\Phi^-(\zeta,\mu)$ can be found. □

In addition to above theorem, we present another theorem about asymptotic equalities of $\Phi^+(\zeta,\mu)$ and $\Phi^-(\zeta,\mu)$ according to variable μ in the following.

Theorem 4.6. *The asymptotic equalities of $\Phi^+(\zeta,\mu)$ and $\Phi^-(\zeta,\mu)$ whenever $|\mu| \to \infty$ are given by*

$$\ln \Phi^+(\zeta,\mu) = \begin{bmatrix} 0 \\ e^{i\mu\zeta} \end{bmatrix} + o(1), \quad \mu \in \overline{\mathbb{C}}_+$$

and

$$\ln \Phi^-(\zeta,\mu) = \begin{bmatrix} e^{-i\mu\zeta} \\ 0 \end{bmatrix} + o(1), \quad \mu \in \overline{\mathbb{C}}_-,$$

respectively.

Proof. Firstly, by using the Riemann Lebesgue Lemma, we have

$$\Phi_1^+(\zeta,\mu) = e^{o(1)}, \quad \mu \in \mathbb{R}, \quad |\mu| \to \infty$$

and

$$\Phi_2^+(\zeta,\mu) = e^{e^{i\mu\zeta}+o(1)}, \quad \mu \in \mathbb{R}, \quad |\mu| \to \infty.$$

Secondly, from Lebesgue Convergence Theorem, we get

$$\Phi_1^+(\zeta,\mu) = e^{o(1)}, \quad \mu \in \mathbb{C}_+, \quad |\mu| \to \infty$$

and

$$\Phi_2^+(\zeta,\mu) = e^{e^{i\mu\zeta}+o(1)}, \ \ \mu \in \mathbb{C}_+, \ \ |\mu| \to \infty.$$

If we combine these results, we prove the first asymptotic. To complete the proof, for $\mu \in \overline{\mathbb{C}}_-$ whenever $|\mu| \to \infty$, we get the second for $\Phi^-(\zeta,\mu)$ in a similar way. □

5. Scattering function of multiplicative Dirac system

In this section, we give the Jost function of (2.1)-(2.2) and examine the asymptotic property of this Jost function. Then, we define the scattering function of this problem.

By applying the boundary condition (2.2) to the Jost solutions $\Phi^+(\zeta,\mu)$ and $\Phi^-(\zeta,\mu)$ given in (4.8) and (4.9), respectively, the Jost functions of (2.1)-(2.2) are obtained as follows

$$\chi_+(\mu) = \frac{\Phi_2^+(0,\mu)}{\Phi_1^+(0,\mu)}, \ \ \mu \in \overline{\mathbb{C}}_+, \tag{5.1}$$

$$\chi_-(\mu) = \frac{\Phi_1^-(0,\mu)}{\Phi_2^-(0,\mu)}, \ \ \mu \in \overline{\mathbb{C}}_-. \tag{5.2}$$

Theorem 5.1. *The Jost functions of (2.1)-(2.2) have the following asymptotic representations whenever* $|\mu| \to \infty$

$$\ln \chi_+(\mu) = 1 + o(1), \ \ \mu \in \overline{\mathbb{C}}_+,$$

$$\ln \chi_-(\mu) = 1 + o(1), \ \ \mu \in \overline{\mathbb{C}}_-.$$

Proof. From (4.8) and (5.1), we write

$$\chi_+(\mu) = e^{\int_0^\infty \left(M(0,s)^{e^{i\mu s}}\right)^{ds}},$$

where $M(0,s) = \dfrac{\Psi_{22}(0,s)}{\Psi_{12}(0,s)}$.

Let first assume that $\mu \in \mathbb{R}$. Then, by the help of Riemann Lebesgue Lemma, we have

$$\chi_+(\mu) = e^{1+o(1)}, \ \ |\mu| \to \infty.$$

Now, let $\mu \in \overline{\mathbb{C}}_+$. Using (5.1) and Lebesgue Convergence Theorem, it is easily seen that

$$\chi_+(\mu) = e^{1+o(1)}, \ \ |\mu| \to \infty.$$

Combination of these last two results completes the first asymptotic. Similarly, for $\mu \in \overline{\mathbb{C}}_-$, the desired asymptotic equation of the Jost function $\chi_-(\mu)$ can be obtained. It completes the proof. □

The scattering function of the multiplicative Dirac system (2.1)-(2.2) is defined as

$$S(\mu) = \chi_-(\mu)^{\frac{1}{\ln \chi_+(\mu)}}, \ \ \mu \in \mathbb{R}.$$

Theorem 5.2. *For* $\mu \in \mathbb{R}$*, the scattering function of (2.1)-(2.2) satisfies the following equality*

$$\lim_{|\mu|\to\infty} S(\mu) = 1.$$

Proof. The proof of Theorem 5.2 is easily seen from Theorem 5.1. □

6. Conclusion

In this study, we discuss the multiplicative Dirac system. We obtain the Jost solution, Jost function and scattering function of this system. It is important because it is the first study in the literature to examine the scattering theory of the multiplicative Dirac system. This study will be a material for researchers who study on scattering theory. Since the Dirac system has great importance in quantum physics in the classical case, it is believed that in the multiplicative case as well, this system will make a significant contribution to mathematical physics. Similar problems can be consider for Schrödinger equations.

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