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RESEARCH ARTICLE

# The Evolution and Deformation of Magnetic Flux Tube Along Neutral Points

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## ABSTRACT

**This study aims to characterize magnetic flux tubes (MFTs) associated with null (neutral) curves in the three-dimensional Lorentzian space. To this end, a system of differential equations is first obtained for the flux surfaces associated with a null curve, using the magnetic vector field components and the Frenet frame at each point. The solution of this system of equations allows us to determine the magnetic field lines and kinematic equations of the magnetic tube surfaces. Furthermore, characteristic values of the Jacobian matrix generated by the magnetic field components are utilized to describe the deformation of the MFTs near null points. A new geometric perspective is then presented through split-quaternion algebra, demonstrating that an MFT can be fully characterized by a split-quaternion product or a homothetic motion. Finally, analytical solutions of the kinematic equations are obtained and illustrated with some theoretical examples.**

***Keywords:*** *Quaternion algebras, mathematical physics, neutral points, magnetic theory, ordinary differential equations.*

***AMS Subject Classification (2020):*** *Primary:53Z04 ; Secondary: 32A07; 53B50; 37C10.*

## 1. Introduction

Quaternions are a crucial instrument for the investigation of motion in physics and geometry. The most significant study on this subject, which defined quaternions, was published in 1844 [37]. Cayley explains the non-commutative nature of quaternion multiplication and presents preliminary ideas regarding the relationship between quaternions and their matrix representation, and subsequently proposes general frameworks for multi-dimensional algebraic structures that will lay the groundwork for the later development of Clifford algebras and Lie groups [9]. The way for the algebraic expression of spacetime geometry, particularly Lorentz transformations, using biquaternions, which are an extended form of Hamilton's quaternion theory with complex numbers are given in [14]. The use of quaternions with vector analysis through geometric applications such as curves, surfaces, and rotations, and highlights their advantages in applied mathematics are obtained in [38]. Hamilton's quaternion theory, its algebraic structure, and its four-component (one scalar, three vectorial) form are detailed, as is their application to classical mechanics, particularly rigid body motion and rotational dynamics, as well as their contributions to analytic geometry and vector analysis [63]. E.Study introduces a new approach to the geometric representation of mechanics and motion using dual quaternions, forming the foundational principles of kinematic analyses used in modern robotics and incorporating approaches that combine differential geometry and algebra [61]. McAulay defines octonions, an eight-component hypercomplex number system, in detail and examines the particularly asymmetric properties of these algebraic structures, demonstrating how octonions differ from classical algebras [51]. Shoemake demonstrates that quaternions provide both computational efficiency and more natural visual transitions for achieving smoother and more stable rotational movements of objects in three-dimensional computer graphics [58]. By addressing the mathematical foundations of quaternions and transformations in space in an understandable manner, Kuipers explains in detail how quaternions are used, especially in space mechanics,

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satellite orbits, aviation, and virtual reality applications [47]. Vicci systematically demonstrates how rotations in three-dimensional space can be modeled by examining the way quaternions represent these rotations both algebraically and geometrically, shedding light on the applicability of quaternions for researchers working in the fields of computer science and microelectronics [67]. Giardino's research integrates differential geometry with quaternion algebra, providing innovative and effective mathematical instruments, particularly for modeling movements and transformations [29]. Synge demonstrates how quaternions can be used to express Lorentz transformations in 4-dimensional Minkowski spacetime, making the mathematical structure of quantum mechanics and relativity theory more symmetric [62]. Rösch mathematically addresses in this article how the Hamiltonian is algebraically analyzed under time-reversal symmetry and how it reflects on eigenvalue problems [**?**]. The hyperbolic rotations in Minkowski 3-space (1+2 dimensional spacetime) using a special class of split quaternions known as unit time-like quaternions to model transformations occurring in Minkowski 3-space and demonstrates their applicability to physical geometry is defined in [55]. It is demonstrated that split quaternions are also a powerful tool for semi-Riemannian differential geometry and projective geometry, explaining the more effective use of quaternions in relativity and field theories [3]. The authors show that the structure of split quaternions is compatible with Minkowski space and can naturally represent physical systems in this space [33]. Frank Antonuccio, presents an alternative formulation of the Dirac equation using split quaternions [1]. Jafari and Yaylı systematically examine the matrix theory established on split quaternions, making the matrix structures based on split quaternions compatible with classical linear algebra, and developing the physical and geometric applications of these structures [40]. Constantly inclined spacelike surfaces defined in Minkowski 3-space are expressed parametrically using the transformation properties of split quaternions, offering an alternative and more structural perspective to classical differential geometry tools [4]. Libine introduces the analytical structure of split quaternions by comparing them with classical complex analysis [50].The authors illustrate that the traditional Maxwell equations can be consolidated into a singular equation through hyperbolic quaternion formalism, so offering a more lucid and geometric representation of electromagnetic field dynamics in spacetime[17]. Gogberashvili's work shows that split quaternions can be directly used in electromagnetic field theories [34]. Duru and Çolakoğlu introduce generalized split quaternions and examine their applications on non-parabolic conic transformations, thereby aiming to provide a new perspective on differential geometry and Lorentz-type transformations [19]. Yang, Wang, and Kou are examines the tensor equations defined by dual split quaternions combining the hyperbolic structure of split quaternions with the differential-geometric advantages of dual numbers [71]. Many researchers have become interested in surface investigation using quaternions, and various surfaces related to this have been investigated in various spaces due to the appropriate quaternions for these spaces [70, 2, 35, 66, 42]. Northrop, by combining with transformation theories such as quaternions or split quaternions, allows for the modeling of particle movements [53]. The topological and physical conservation laws of magnetic helicity in the modeling of magnetic fields using quaternionic and split-quaternionic formulations are presented in [8]. Francis F. Chen's classic work serves as an introduction to plasma physics and also explains the circular (spiral) motions of charged particles in magnetic fields [13]. This article is one of the first systematic studies for the 3D kinematic reconnection theory, explaining how reconnection occurs in structures such as closed field lines and null points [48]. This study demonstrates the combination of energy-minimization principles of magnetic fields with topological constraints [15]. It is demonstrated that quaternions naturally arise as a mathematical language in fluid mechanics and plasma physics equations [25]. The authors first theoretically predict the magnetic compression processes in hyperbolic magnetic flux tubes (HFT) and then validate their dynamic evolution with three-dimensional MHD simulations [64, 24]. Moreover, Barros published an important publication related to geometry and the Landau-Hall problem [5]. In their studies, Gibbon and colleagues express particle dynamics in Euler fluid equations using quaternions and have developed a powerful method for the geometric analysis of orientation and rotational motion in fluids [27, 28]. The authors use the Frenet-Serret frame to explicitly relate notions like magnetic helicity and vortex topology to differential geometry in the context of magnetic field line geometric evolution [6]. A significant book discussing the ramifications of mathematical methods in the study of particles and other physical structures has recently been published [69]. This research explores the application of plasma and magnetohydrodynamic (MHD) dynamics to relativistic or high-energy systems through quaternionic formulations [26, 10, 11, 12]. Furthermore, the quaternion approach is used to evaluate flows [20], and flux tube surfaces are studied theoretically and practically with the aid of quaternions [72, 54]. Along with investigating electromagnetic theory with differential calculus [68] and kinematic approach for magnetic theory [49].This study by Giardino employs quaternion algebra to express classical electromagnetism equations and Maxwell's equations in a compact, symmetric, and naturally geometric framework[30]. Curves are expanded by three names as metric-operated curves in the Lorentz-Minkowski space. These curves include null curves, which are good candidates for studying physical components. Curves and surfaces,

two of the most crucial geometrical elements, and the relationships between all of their structural elements have been studied in order to evaluate the motion of the particle in magnetic theory. In space, light moves along a null curve. Consequently, it has become crucial to look into null curves in order to look at how light moves [18, 39, 36].Fernandez, Gimenez, and Lucas , have examined curve theory in the context of magnetic fields from both geometric and physical perspectives [21, 22], while Kazan and Karadağ have investigated magnetic pseudo-null and null curves in Minkowski 3-space [41], and Özdemir has analyzed the mathematical properties of null Cartan helical trajectories in Lorentzian 3-space [56]. Nešović examines rotation motions around a light-like axis [52], while Şimşek and Özdemir formulate hyperbolic rotation matrices in Minkowski 3-space and rotations on the light cone using Clifford algebra and linear algebra methods [59, 60]. Recently, flux tube surfaces and magnetic flux surfaces have been the subject of numerous studies. Numerous mathematicians and physicists are interested in this multidisciplinary topic, and research on it has accelerated [43, 44, 45, 31, 32, 7, 46, 65]. Around the magnetic field's neutral points, the magnetic field line has several characterizations. Surfaces associated with these magnetic field lines undergo topological alterations as a result of the reconnected magnetic field lines around these points. The motivation of this article is to generate surfaces that accept the magnetic field lines surrounding neutral points as parameter curves to visualize the shape change on the surface. Quaternions are one of the best tools for characterizing these kinds of surface. Here, the quaternion is a timelike quaternion with a null vector part. Two methods are examined separately when generating flux tube surfaces around neutral points. In the first method, the magnetic field is first determined, and flux tube surfaces are generated with eq.(22) with the help of the $\alpha$ curve, which is the integral curve of this magnetic field and consists of neutral points, and the parabola curve (the trajectory followed by a point rotating around a null axis). Secondly, the magnetic field components are obtained with the help of the Cartan frame, and a system of equations is obtained. Then, a magnetic flux tube is obtained by solving this system of equations. Characterization of all flux tube surfaces near neutral points is given with the help of the characteristic values of the matrix form corresponding to the magnetic field vector. Undoubtedly, quaternion algebra that helps us make our calculations and visualization simpler and easier in this study.

In this study, we used geometry and kinematics to analyze the flux tubes around the neutral points. In the first section, we have discussed the physics and mathematics studies and the historical context that we employed in this study. The mathematical terms and terminology that we have used throughout this research are provided in the second section. The flux tube equation constructed along the neutral point is derived and its components are obtained in the third section. We investigate the flux tube motion along this neutral point using the quaternion technique in the fourth chapter. We also studied the magnetohydrodynamics equations and Maxwell's equations from a quaternionic viewpoint. In the fifth section, we examine the deformations and variation of these flux tubes near the neutral points and found some extremely fascinating geometrical findings. With the aid of a computer program, we visualized these flux tubes in the sixth chapter. Physical and mathematical results are presented in the final section.

## 2. Basic concepts and background

Let $u = (u_1, u_2, u_3)$, $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ and $a_1, a_2, a_3 \in \mathbb{R}^+$. After that, $g : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$g(u, w) = -a_1 u_1 w_1 + a_2 u_2 w_2 + a_3 u_3 w_3 \tag{2.1}$$

is an indefinite scalar product that is called a *Lorentzian g-inner product*. We can also state the g-inner product as the matrix form $g(u, w) = u^t \Omega w$ in which there is a matrix. $\omega$ is given by

$$\begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

If $g(u, u) > 0$, $g(u, u) = 0$ for $u \neq 0$, or $g(u, u) < 0$, then the vector $u$ is said to be spacelike, null (lightlike, isotropic), or timelike, respectively. The following three forms can be derived for the pseudo-spheres of $R_g^{1,2}$:

$$S_g^2 = \{u \in R_g^{1,2} | g(u, u) = 1\}, \tag{2.2}$$
$$H_g^2 = \{u \in R_g^{1,2} | g(u, u) = -1\}, \tag{2.3}$$
$$LC_g^2 = \{u \in R_g^{1,2} | g(u, u) = 0\}. \tag{2.4}$$

They are referred to as, respectively, the g-de Sitter 2-space, g-hyperbolic 2-space, and g-light cone. Given two vectors $u$ and $w$ in the same timelike cone, the hyperbolic angle between them as follows

$$\cos\theta = \frac{-g(u,w)}{\|u\|_g \|w\|_g}. \tag{2.5}$$

The definition of the g-vector product of two vectors $\{u,w\} \in \mathbb{R}^{1,2}_g$ is defined as

$$v_g(u \times w) = \Delta det \begin{bmatrix} -e_1/a_1 & e_2/a_2 & e_3/a_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{bmatrix}, \tag{2.6}$$

where $\Delta = \sqrt{|det\Omega|}$. Let $R \in \mathbb{R}^3_3$ be any $3\times 3$-matrix such that $g(Ru,Rw) = g(u,w)$ for all vectors $\{u,w\} \in \mathbb{R}^3$, then $R$ is referred to as a semi-$g$-orthogonal matrix form. The set of semi-$g$-orthogonal matrices is shown by the symbol $SO_g(1,2)$ and $g$ rotation matrix satisfies $R^t\Omega R = \Omega$, $detR = 1$, namely given by

$$SO_g(1,2) = \{R : R^t\Omega R = \Omega \; and \; detR = 1, R, 3\times 3 \; matrix\}. \tag{2.7}$$

Let us take the $\Delta^* = \sqrt{a_1a_2a_3}$ and $g$-skew symmetric matrix given by

$$A = \Delta^* \begin{bmatrix} 0 & u_3/a_1 & -u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix} \tag{2.8}$$

then we have $p(x) = x^3 - < u,u > x$, $A^3 = 0$, since $u$ is a null vector, [52].

The basis $\{L,N,W\}$ in $\mathbb{R}^{1,2}$ defined as a pseudo-orthonormal $g$-frame and followings are satisfied

$$\partial_s L = k_1 W, \tag{2.9}$$
$$\partial_s N = k_2 W, \tag{2.10}$$
$$\partial_s W = -k_2 L - k_1 W, \tag{2.11}$$

where $k_1 = \sqrt{g(\partial_s^2\alpha, \partial_s^2\alpha)}$, $k_2 = g(N,W)$ are curvatures, and we have

$$g(L,L) = g(N,N) = g(L,W) = g(N,W) = 0, g(L,N) = g(W,W) = 1, \tag{2.12}$$
$$v_g(N\times L) = W, v_g(L\times W) = L, v_g(W\times N) = N. \tag{2.13}$$

**Theorem 2.1.** *Let $A$ be a semi-$g$-skew symmetric matrix form with respect to this formula (2.8) and let $u$ be a null vector which is $u \in span(N)$ where $N = (u_1,u_2,u_3)$. Then the matrix*

$$R^{g,u}_\phi = e^{-A\phi} = I - \phi A + \frac{\phi^2}{2}A^2, \tag{2.14}$$

*gives a $g$ rotation matrix about the null axis $u$. The matrix represents a rotation motion on the g-lightlike cone $a_1x^2 = a_2y^2 + a_3z^2$ using the equation (2.8) for $A$ it can be expressed as*

$$\begin{bmatrix} \frac{1}{2}\phi^2(a_2u_2^2 + a_3u_3^2) + 1 & -\frac{\phi}{a_1}u_3\Delta^* - \frac{1}{2}\phi^2 a_2u_1u_2 & \frac{\phi}{a_1}u_2\Delta^* - \frac{1}{2}\phi^2 a_2u_1u_2 \\ \frac{1}{2}\phi^2 a_1u_1u_2 + a_3u_3^2 - \frac{\phi}{a_2}u_3\Delta^* & 1 - \frac{1}{2}\phi^2(a_1u_1^2 - a_3u_3^2) & \frac{\phi}{a_2}u_1\Delta^* - \frac{1}{2}\phi^2 a_3u_2u_3 \\ \frac{\phi}{a_3}u_2\Delta^* + \frac{1}{2}\phi^2 a_1u_1u_3 & -\frac{\phi}{a_3}u_1\Delta^* - \frac{1}{2}\phi^2 a_2u_2u_3 & 1 - \frac{1}{2}\phi^2(a_1u_1^2 - a_2u_2^2) \end{bmatrix}. \tag{2.15}$$

Moreover, the rotation motion about the null axis can be easily expressed by the hyperbolic split quaternion algebra. The hyperbolic split quaternion space denoted by $\widehat{H}_{a_1,a_2,a_3}$ and the base elements satisfy $i^2 = -a_1$, $j^2 = a_2$, $k^2 = a_3$ $ij = \frac{\Delta^*}{a_3}k = -ji$, $jk = \frac{\Delta^*}{a_1}i = -kj$, $ki = \frac{\Delta^*}{a_2}j = -ik$, where $\Delta^* = \sqrt{a_1a_2a_3}$, $a_1,a_2,a_3 \in R^+$. Then a hyperbolic split quaternion is given by

$$q = q_0 + a_1i + a_2j + a_3k = s_q + v_q, \; where \; s_q = q_0 \; and \; v_q = a_1i + a_2j + a_3k. \tag{2.16}$$

The quaternion product is given as

$$pq = p_0q_0 + g(v_p, v_q) + p_0v_q + q_0v_p + v_g(v_p \times v_q). \tag{2.17}$$

If $p, q$ pure quaternions, then we have

$$pq = g(v_p, v_q) + v_g(v_p \times v_q) \tag{2.18}$$

$$= -a_1p_1q_1 + a_2p_2q_2 + a_3p_3q_3 + \Delta^* \begin{vmatrix} -i/a_1 & j/a_2 & k/a_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}. \tag{2.19}$$

Thus, we get

$$pq = \begin{bmatrix} p_0 & -a_1p_1 & a_2p_2 & a_3p_3 \\ p_1 & p_0 & \frac{p_3\Delta^*}{a_1} & \frac{-p_2\Delta^*}{a_1} \\ p_2 & \frac{p_3\Delta^*}{a_2} & p_0 & \frac{-p_1\Delta^*}{a_2} \\ p_3 & \frac{-p_2\Delta^*}{a_3} & \frac{p_1\Delta^*}{a_3} & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \tag{2.20}$$

For any hyperbolic split quaternion $q = q_0 + q_1i + q_2j + q_3k$, the conjugate, norm, and inverse are defined as follows, respectively

$$\overline{q} = q_0 - q_1i - q_2j - q_3k = s_q - v_q,$$

$$N_q = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{|q_0^2 + a_1q_1^2 - a_2q_2^2 - a_3q_3^2|},$$

$$q^{-1} = \frac{\overline{q}}{J_q}, J_q \neq 0, J_q = q_0^2 + a_1q_1^2 - a_2q_2^2 - a_3q_3^2.$$

The hyperbolic split quaternions are called three characters that are $H$-spacelike, $H$-timelike, or $H$-null if $J_q > 0$, $J_q < 0$, or $J_q = 0$, respectively, [59].
In this study, we examined rotation along the null axis w.r.t. timelike hyperbolic split quaternion, so we give some valuable information about this case. The set of $H$-timelike quaternions is denoted as follows:

$$TH_{a_1,a_2,a_3} = \{q = (q_1, q_2, q_3, q_4) : q_1, q_2, q_3, q_4 \in \mathbb{R}, J_q > 0\}.$$

**Theorem 2.2.** *Each $H$-timelike unit hyperbolic split quaternion that can be written as $q = 1 - \frac{\phi}{2}\boldsymbol{u}$ according to the $\boldsymbol{u}$ is a null vector part of this quaternion. Then, in this case, this quaternion rotates, namely, g-rotation, about a null axis along $\phi \in \mathbb{R}$ [60].*

Moreover, we assume that $q$ and $p$ are two $H$-timelike unit hyperbolic split quaternions. So, it can be written as follows;

$$R_q(R_p(v)) = qR_p(v)\overline{q} = R_{qp}(v)$$

where $R_p$ and $R_q$ are $g$-rotation matrices and v is an arbitrary vector [60].

**Theorem 2.3.** *Let $p$ and $q$ be two unit $H$-timelike hyperbolic split quaternion that are defined as follow,*

$$p = 1 - \frac{\phi_1}{2}\boldsymbol{u}, \quad q = 1 + \frac{\phi_2}{2}\boldsymbol{u}$$

*that have same null vector part $\boldsymbol{u}$. Then, $R_{pq}$ rotates about null axis through the angle $\phi_1 + \phi_2$ where $\phi_1, \phi_2 \in \mathbb{R}$ [60].*

## 3. Magnetic field components of flux tube

The magnetic flux tube surface along the neutral (null) point can be expressed as follows,

$$\alpha(s,t) = \alpha(s) + r(s)(tN(s) + \frac{t^2}{2}W(s)). \tag{3.1}$$

Let $D = (e_1, e_2, e_3)$ be $g$-frame along a curve then we have following two matrix

$$U = \Delta^* \begin{bmatrix} 0 & u_3/a_1 & -u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix}$$

and

$$W = \Delta^* \begin{bmatrix} 0 & w_3/a_1 & -w_2/a_1 \\ w_3/a_2 & 0 & -w_1/a_2 \\ -w_2/a_3 & w_1/a_3 & 0 \end{bmatrix},$$

where $\Delta^* = \sqrt{a_1 a_2 a_3}$ and the matrices $\{U, W\}$ are g-skew symmetric matrices.

To define the flux tube, we need a parabola $(0, t, t^2)$ and a null space curve $\alpha$. The frame on the parabola is denoted by the Frenet frame apparatus $\{t, n, b\}$ (since the parabola is a spacelike curve). The unit tangent vector $L$ defined by $\nabla_t^* \alpha(s,t) = D\xi = \xi L$ along the curve $\alpha(s,0)$, here $D\xi = -a_1\xi_1 t + a_2\xi_2 n + a_3\xi_3 b$ is $g$ product, $v \in \mathbb{R}_g^{1,2}$, $\nabla_s^* D = DU$, and $\nabla_t^* D = DW$. The connection $\nabla^*$ on $\mathbb{R}_g^{1,2}$ satisfy $\nabla_w^* x_i = \frac{1}{a_i}\partial_w x_i$, where $\partial_w$ is standard Euclidean derivative.

Additionally, we can state that the magnetic vector field $B$ along the flux tube is provided by the tangent vectors of the $s$ parameter curves, namely the space curve $\alpha$ is an integral curve of the magnetic field $B$, as follows:

$$\nabla_s^* \alpha(s,t) = B(s,t) = DB, B = (B_1, B_2, B_3). \tag{3.2}$$

If we use the derivatives w.r.t $t$ and $s$ parameters, we get the following representations:

$$\nabla_s^*(\nabla_t \alpha) = \nabla_t^*(\nabla_s^* \alpha), \nabla_s^* D = DU, \tag{3.3}$$

$$\nabla_s^*(\nabla_t D) = \nabla_t^*(\nabla_s^* D), \nabla_t^* D = DW. \tag{3.4}$$

Then we obtain

$$\Omega(UB + \nabla_s^* B) = \Omega(w\xi + \nabla_t^*). \tag{3.5}$$

Expressing the resulting equation in terms of components, we have

$$\begin{aligned}
&(\Delta^*)^2 \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} 0 & u_3/a_1 & -u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & -w_1/a_1 & -w_2/a_1 \\ w_3/a_2 & 0 & -w_1/a_2 \\ -w_2/a_3 & w_1/a_3 & 0 \end{bmatrix} \\
&\quad + \Delta^* \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} 0 & \partial_s w_1/a_1 a_3 & -\partial_s w_2/a_1 a_2 \\ \partial_s w_3/a_2 a_3 & 0 & \partial_s w_1/a_1 a_2 \\ -\partial_s w_2/a_3 a_2 & \partial_s w_1/a_3 a_1 & 0 \end{bmatrix} \\
&= (\Delta^*)^2 \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} 0 & w_3/a_1 & -w_2/a_1 \\ w_2/a_2 & 0 & -w_1/a_2 \\ -w_2/a_3 & w_1/a_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & u_3/a_1 & -u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix} \\
&+ \Delta^* \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} 0 & \partial_t u_3/a_1 a_3 & -\partial_t u_2/a_1 a_2 \\ \partial_t u_3/a_2 a_3 & 0 & -\partial_t u_1/a_1 a_2 \\ -\partial_t u_2/a_3 a_2 & \partial_t u_1/a_3 a_1 & 0 \end{bmatrix} \\
&\quad \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \left( \Delta^* \begin{bmatrix} 0 & u_3/a_1 & -u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} + \begin{bmatrix} \partial_s B_1/a_1 \\ \partial_s B_2/a_2 \\ \partial_s B_3/a_3 \end{bmatrix} \right) \\
&\quad = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \left( \Delta^* \begin{bmatrix} 0 & w_3/a_1 & -w_3/a_1 \\ w_3/a_2 & 0 & -w_1/a_2 \\ -w_2/a_3 & w_1/a_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\partial + \xi}{a_3} \end{bmatrix} \right).
\end{aligned}$$

Thus, we can calculate the following differential equation systems:

$$\frac{\partial_s B_1}{\Delta^*} + u_2 B_3 - u_3 B_2 = w_2, \tag{3.6}$$

$$\frac{\partial_s B_2}{\Delta^*} + u_3 B_1 - u_1 B_3 = -w_1, \tag{3.7}$$

$$\frac{\partial_s B_3}{\Delta^*} + u_1 B_2 - u_2 B_1 = \partial_t, \tag{3.8}$$

$$\partial_t u_3 - \partial_s w_3 = u_2 w_1 - u_1 w_2, \tag{3.9}$$

$$\partial_t u_1 - \partial_s w_1 = u_3 w_2 - u_2 w_3, \tag{3.10}$$

$$-\partial_t u_2 + \partial_s w_2 = u_1 w_3 - u_3 w_1. \tag{3.11}$$

Thus, we can give the following theorems.

**Theorem 3.1.** *Let $B$ be a magnetic field and $\alpha$ be a null magnetic field line associated with $B$. After that, a magnetic flux tube along the curve $\alpha$ can be constructed by the solution of the following equation systems:*

$$\frac{\partial_s B_1}{\Delta^*} + u_2 B_3 - u_3 B_2 = w_2, \tag{3.12}$$
$$\frac{\partial_s B_2}{\Delta^*} + u_3 B_1 - u_1 B_3 = -w_1, \tag{3.13}$$
$$\frac{\partial_s B_3}{\Delta^*} + u_1 B_2 - u_2 B_1 = \partial_t, \tag{3.14}$$
$$\partial_t u_3 - \partial_s w_3 = u_2 w_1 - u_1 w_2, \tag{3.15}$$
$$\partial_t u_1 - \partial_s w_1 = u_3 w_2 - u_2 w_3, \tag{3.16}$$
$$-\partial_t u_2 + \partial_s w_2 = u_1 w_3 - u_3 w_1. \tag{3.17}$$

*which gives a system of differential equations of magnetic field velocity components $B = \{B_1, B_2, B_3\}$, then the integration of the equation $\nabla_s^* \alpha(s,t) = DB$ gives the related tube surfaces.*

Next, we will give a theorem that determines the magnetic field velocity components $B = \{B_1, B_2, B_3\}$ of the MFT.

**Theorem 3.2.** *Let us assume that $B$ is a magnetic field and $\alpha$ is a null magnetic field line associated with $B$. After that, the magnetic field line velocity components $\{B_1, B_2, B_3\}$ of MFT satisfy*

$$B_1 = \frac{(\sqrt{2}+1)(r(s)t)}{\sqrt{2+2t^2}}, \tag{3.18}$$
$$B_2 = \frac{r(s)(-1+\sqrt{2}t^2)}{\sqrt{2+2t^2}}, \tag{3.19}$$
$$B_3 = \frac{r(s)}{\sqrt{2}}. \tag{3.20}$$

where $r(s)$ stretch function of the magnetic flux tube $\alpha(s,t)$.

*Proof.* We can define the flux tube along the null curve as follows: $\alpha(s,t) = \alpha(s) + r(s)(tN(s) + \frac{t^2}{2}W(s))$. If $\alpha$ is a null curve then we have the following pseudo-orthonormal $g$-frame

$$\partial_s L = k_1 W, \tag{3.21}$$
$$\partial_s N = k_2 W, \tag{3.22}$$
$$\partial_s W = -k_2 L - k_1 W, \tag{3.23}$$

where $k_1 = \sqrt{g(\partial_s^2\alpha, \partial_s^2\alpha)}$, $k_2 = g(N,W)$. The cross-section curve ($t$-parameter curves) that moves along the magnetic field line ($s$-parameter curves) of the flux tube is defined as $\gamma = (0, t, \frac{t^2}{2})$. Since $\partial_t\gamma = (0,1,t)$, $||\partial_t\gamma||_g = \sqrt{1+t^2} > 0$, the cross-section curve of the flux tube is a spacelike curve. We can write the cross-section curve $\gamma$ of the flux tube w.r.t pseudo-orthonormal $g$-frame as follows:

$$\gamma = tW + \frac{t^2}{2\sqrt{2}}(L+N). \tag{3.24}$$

Differentiating the last equation with respect to $t$-parameter, we reach

$$\partial_t\gamma = W + \frac{t}{\sqrt{2}}(L+N). \tag{3.25}$$

Thus, we compute

$$||\gamma'||_g = \sqrt{1+t^2}. \tag{3.26}$$

The formula for the tangent vector of the curve $gamma$ is

$$\mathbf{t} = \frac{W}{\sqrt{1+t^2}} + \frac{t}{\sqrt{1+t^2}}(\frac{L}{\sqrt{2}} + \frac{N}{\sqrt{2}}). \tag{3.27}$$

If we compute the second derivative of the curve $\gamma$, we obtain

$$\gamma'' = \frac{L}{\sqrt{2}} + \frac{N}{\sqrt{2}}. \tag{3.28}$$

and

$$v_g(\gamma' \times \gamma'') = \frac{L}{\sqrt{2}} - \frac{N}{\sqrt{2}}. \tag{3.29}$$

These give the binormal vector field of $\gamma$ as follows:

$$\mathbf{b} = \frac{L-N}{\sqrt{2}}. \tag{3.30}$$

Then, we have

$$\mathbf{n} = v_g(\mathbf{b} \times \mathbf{t}) = -\frac{(L+N)}{\sqrt{2}\sqrt{1+t^2}} + \frac{tW}{\sqrt{1+t^2}}. \tag{3.31}$$

However, we can also write

$$\alpha_t(s,t) = B_1\mathbf{t} + B_2\mathbf{n} + B_3\mathbf{b} \tag{3.32}$$

where $B = \{B_1, B_2, B_3\}$ magnetic field line. If we take a derivative $\alpha(s,t)$ w.r.t parameter $t$, we can write as

$$\partial_t\alpha(s,t) = B_1\mathbf{t} + B_2\mathbf{n} + B_3\mathbf{b}. \tag{3.33}$$

So, the elements that make up the magnetic field line can be calculated using this formula

$$B_1 = g(\alpha_t(s,t), \mathbf{t}), \tag{3.34}$$
$$B_2 = g(\alpha_t(s,t), \mathbf{n}), \tag{3.35}$$
$$B_3 = -g(\alpha_t(s,t), \mathbf{b}). \tag{3.36}$$

The derivative of the flux tube is computed by

$$\partial_t\alpha(s,t) = r(s)(N(s) + tW(s)). \tag{3.37}$$

Thus, magnetic field line elements that make up the flux tube's magnetic field line are calculated as

$$B_1 = \frac{(\sqrt{2}+1)(r(s)t)}{\sqrt{2+2t^2}}, \tag{3.38}$$
$$B_2 = \frac{r(s)(-1+\sqrt{2}t^2)}{\sqrt{2+2t^2}}, \tag{3.39}$$
$$B_3 = \frac{r(s)}{\sqrt{2}}. \tag{3.40}$$

□

## 4. Magnetic flux tube along neutral points with quaternions

The following conservation laws, which are derived from the electromagnetic and fluid mechanics equations, are embodied in the magnetohydrodynamics equations.
*1. Faraday's equation:*

$$\frac{\partial B}{\partial t} = -c\nabla \times E$$

*2. Ampere's law:*

$$\nabla \times B = J\mu_0$$

*3. Gauss law:*

$$\nabla \cdot B = 0$$

*4. Ohm's law:*

$$E + v \times B = \eta J,$$

in which, respectively, the electric field $E$, the velocity $v$, the magnetic field $B$, the current density $J$ and the vacuum permeability $\mu_0$ are in [24]. The following are generalized Dirac-Maxwell equations:

$$\begin{aligned}
\nabla \cdot E &= \rho^e, \\
\nabla \cdot B &= \rho^m, \\
\nabla \times E &= -\frac{\partial B}{\partial t} - J^m, \\
\nabla \times B &= \frac{\partial E}{\partial t} + J^e,
\end{aligned}$$

where the magnetic charge and current densities are $\rho^m$ and $J^m$ , while $\rho^e$ and $J^e$ are the electric charge and current densities.

$$\begin{aligned}
\sigma^e(E + v \times B) &= J^e, \\
\sigma^m(E + v \times B) &= J^m,
\end{aligned}$$

where $\sigma^m$ is the magnetic conductivity.We have an explanation for the dyons' infinite conductivity,

$$E = -(v \times B) \text{ and } B = (v \times E),$$

[11]. The following are the magnetohydrodynamics equations' quaternionic representations:

$$\begin{aligned}
\nabla \cdot E &= \frac{\rho^e}{\varepsilon}, \ (\textit{Imaginary part of } e_0) \\
\nabla \cdot B &= \mu\rho^m, \ (\textit{Real part of } e_0) \\
(\nabla \times E)_x &= -\frac{\partial B_x}{\partial t} - \frac{J_x^m}{\varepsilon}, \ (\textit{Imaginary part of } e_1) \\
(\nabla \times E)_y &= -\frac{\partial B_y}{\partial t} - \frac{J_y^m}{\varepsilon}, \ (\textit{Imaginary part of } e_2) \\
(\nabla \times E)_z &= -\frac{\partial B_z}{\partial t} - \frac{J_z^m}{\varepsilon}, \ (\textit{Imaginary part of } e_3) \\
(\nabla \times B)_x &= \frac{1}{a_0^2}\frac{\partial E_x}{\partial t} + \mu J_x^e, \ (\textit{Real part of } e_1) \\
(\nabla \times B)_y &= \frac{1}{a_0^2}\frac{\partial E_y}{\partial t} + \mu J_y^e, \ (\textit{Real part of } e_2) \\
(\nabla \times B)_z &= \frac{1}{a_0^2}\frac{\partial E_z}{\partial t} + \mu J_z^e, \ (\textit{Real part of } e_3)
\end{aligned}$$

where, $a_0 = \frac{1}{\mu\epsilon}$, $\rho^e$ and $J^e = q^e n^e v^e$ are, respectively, the electric charge and current densities, $\rho^m$ and $J^m = q^m n^m v^m$ are magnetic charge and current densities. The eight equations mentioned above can be used to represent the quaternionic field equations for dyonic cold plasma hydrodynamics. These equations are the fundamental ones for MHD and are identical to Dirac-Maxwell equations in general (see [11])

The surface of MFT parallel to the neutral point can be expressed as

$$\alpha(s,t) = \alpha(s) + r(s)(tN(s) + \frac{t^2}{2}W(s)). \tag{4.1}$$

If we take a unit quaternion $q = 1 - \frac{t}{2}L$, we compute

$$\begin{aligned}
\alpha(s,t) &= \alpha(s) + tr(s)(1 - \frac{t}{2}L)N \\
&= \alpha(s) + r(s,t)qN,
\end{aligned} \tag{4.2}$$

where $r(s,t) = r(s)t$. As a result, we arrive at the following theorem without providing a proof.

**Theorem 4.1.** *Let us take a null curve $\alpha$ and a unit quaternion $q = 1 - \frac{t}{2}L$, then the quaternionic representation of the flux tube surface $\alpha(s,t)$ along the null curve $\alpha$ can be given as follows*

$$\alpha(s,t) = \alpha(s) + r(s,t)qN, \tag{4.3}$$

*where $r(s,t) = r(s)t$ the stretch function of the flux tube $\alpha(s,t)$.*

When we compute the derivative of the curve $\alpha$ for the $t$ parameter, we get

$$\partial_t \alpha(s,t) = r(s)qN(s). \tag{4.4}$$

In light of this, the magnetic field line components of the MFT along the neutral points are obtained as follows,

$$B_1 = \frac{r(s)q(s,t)}{\sqrt{2+2t^2}}, \tag{4.5}$$

$$B_2 = -\frac{r(s)q(s,t)}{\sqrt{2+2t^2}}, \tag{4.6}$$

$$B_3 = -\frac{r(s)q(s,t)}{\sqrt{2}}, \tag{4.7}$$

where $r(s)$ is the stretch factor function. The following theorem can be used to determine the quaternionic representation of MFT surface corresponding to $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ along the neutral point.

**Theorem 4.2.** *Let $\alpha$ be the magnetic field line associated with $B$ and $q = 1 - \frac{t}{2}L$ be a unit quaternion. Assume that MFT $\alpha(s,t)$ generated with the cross-section curve $\gamma$ and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be Frenet frame's vector fields through the curve $\gamma$. Then the MFT along the curve $\alpha$ can be written as Frenet frame apparatus of the cross-section curve and the unit quaternion $q$ as follows:*

$$\alpha(s,t) = \int \left(\frac{r(s)q(s,t)}{\sqrt{2+2t^2}}\mathbf{t} - \frac{r(s)q(s,t)}{\sqrt{2+2t^2}}\mathbf{n} - \frac{r(s)q(s,t)}{\sqrt{2}}\mathbf{b}\right)dt, \tag{4.8}$$

*where $r(s)$ is the stretch factor function.*

## 5. Evolutions of the magnetic flux tube near the neutral points

According to the following differential equation, magnetic field lines

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}.$$

Two flux surfaces are produced by the Eq.(5)'s solution. It can also be obtained as an intersection curve. If these two flux surfaces $u(x,y,z) = C$ and $v(x,y,z) = K$, then the electric potential $\Phi$ is constant and defined by $\Phi = \Phi(C,K)$. Since $E = -\nabla\Phi$ from here, $E$ is also determined. Thus, the null points of the magnetic field $B$ are obtained thanks to the singularity of $E$ or the equation $v\perp = \frac{E\times B}{\langle B,B\rangle}$. Therefore, the neutral point of any magnetic field is defined as the point where the resulting magnetic field is always zero [48]. In this study, we describe the magnetic flux tube formed around the curve defined as the null curve created by these neutral points. The magnetic field's actions near these neutral points can be determined with the help of eigenvalues corresponding to the matrix form of the magnetic field. The behavior of the magnetic field near the neutral points is explained by the following cases of eigenvalues. Let us give the 3 real matrix form of $B$ as follows.

$$\Delta = \begin{bmatrix} -\frac{\partial B_1}{\partial x_1} & \frac{\partial B_1}{\partial x_2} & \frac{\partial B_1}{\partial x_3} \\ -\frac{\partial B_2}{\partial x_1} & \frac{\partial B_2}{\partial x_2} & \frac{\partial B_2}{\partial x_3} \\ -\frac{\partial B_3}{\partial x_1} & \frac{\partial B_3}{\partial x_2} & \frac{\partial B_3}{\partial x_3} \end{bmatrix}. \tag{5.1}$$

The trace of $B$ must be zero; from here we get $-\frac{\partial B_1}{\partial x_1}+\frac{\partial B_2}{\partial x_2}+\frac{\partial B_3}{\partial x_3}=0$. Let the eigenvalues of the matrix $\Delta$ be $r_1, r_2$ and $r_3$; in this case, $-r_1+r_2+r_3=0$ is found. This gives the following facts about the behavior of $B$ near the neutral points:

1. While maintaining generality and taking the eigenvalue $r_1=0$, we obtain the following two cases:

$i$.If $r_2$ and $r_3$ are real, the magnetic field lines of MFT approaching the neutral point are in the form of a hyperbola curve.

$ii$. If $r_2$ and $r_3$ are complex, then the magnetic field lines of MFT approaching the neutral point are circular. If $r_2=0$ (or $r_3=0$), then $r_1=r_3$ and the magnetic field lines of MFT approaching the neutral point are straight.

2. If all three nonzero eigenvalues $r_1, r_2, r_3$ are real, then the signs of the eigenvalues are $(-,-,-)$ or $(+,+,+)$. In this case, the magnetic field lines of MFT approaching the neutral point are as follows.

$i$. If $r_1<0$, then the magnetic field lines of the MFT approaching the neutral point are in the form of a negative hyperbola curve.

$ii$. If $r_1>0$, the magnetic field lines of the MFT approaching the neutral point are in the form of a positively oriented hyperbola curve.

While maintaining generality, taking the eigenvalue $r_1$ is real and the other two eigenvalues are complex. In this situation,

$i$. If $r_1<0$, then the magnetic field lines of the MFT approaching the neutral point are in the form of a negatively oriented spiral curve.

$ii$. If $r_1>0$, the magnetic field lines of the MFT approaching the neutral point are in the form of a positively oriented spiral curve.

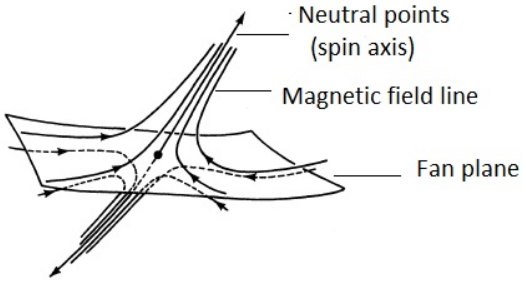



**Figure 1.** A characterization of the neutral points of the magnetic field lines, (see [48]).

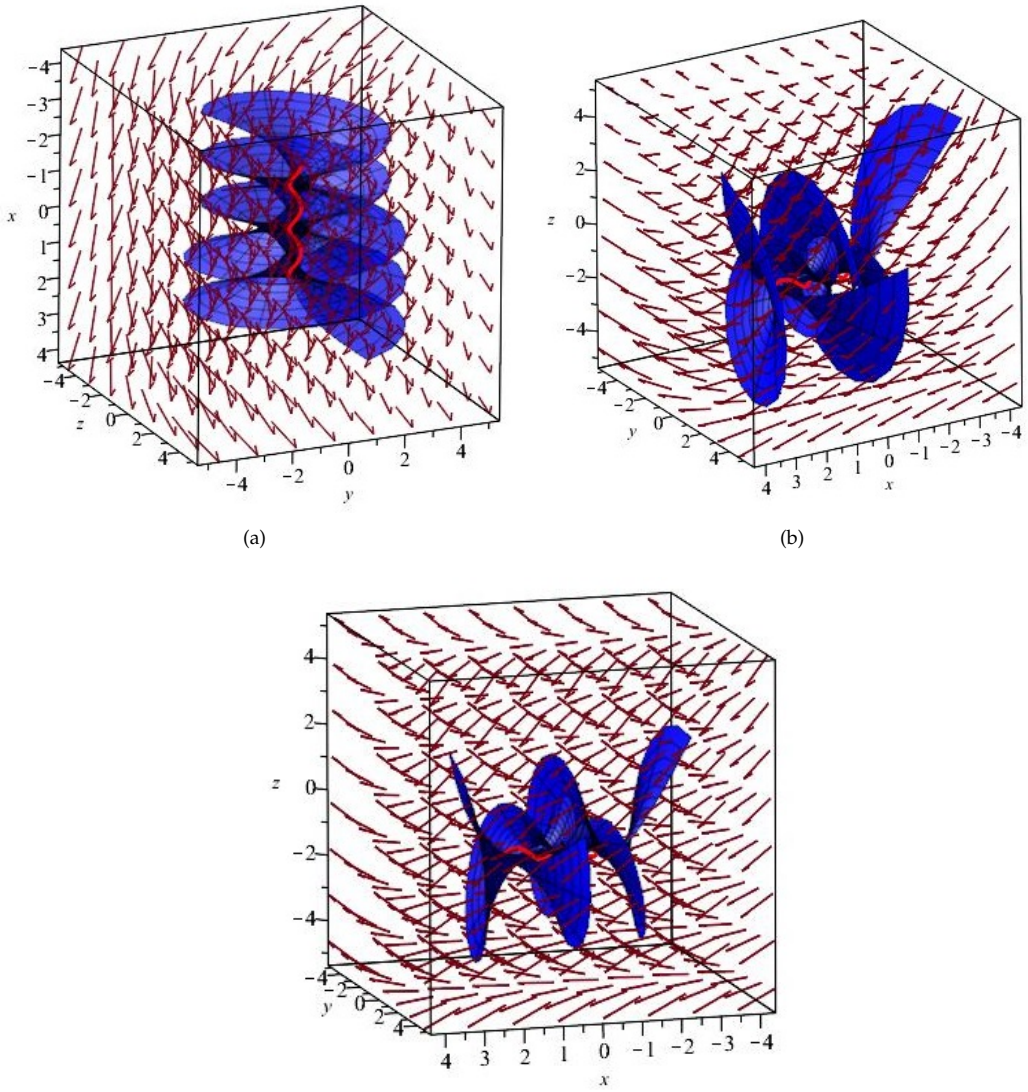
In Figure 1, the fan plane is the plane where the magnetic field lines converge and propagate. Rotation curves are curves that describe the magnetic field lines. That is, they are integral curves of the magnetic field $B$.

Magnetic field lines rotate and approach the neutral points. In this case, MFT, which determines the magnetic field line as its parameter curve, deforms as it approaches the neutral point. We can say that MFT around the neutral point given in this article shows the deformation of MFT formed by the relevant magnetic field lines. For example, the MFT given in Figure 2 shows the deformed magnetic tube surface of the flux tube in the rotational magnetic field line, and the flux tube given in Figure 3 shows the deformed magnetic tube surface in the hyperbolical magnetic field line.

## 6. Applications

In this section, we take $\alpha$ as the magnetic field line and $B$ as the magnetic field.

**Example 6.1.** Let us take $B=(1,5z,-5y)$,so we can calculate $\alpha(s)$ via $\partial_s\alpha=B$ as $\alpha(s)=(s,\frac{1}{5}\sin(5s+4),-\frac{1}{5}\cos(5s+4))$. Namely, we calculate the null integral curves, which consist of neutral points of $B$. Thus, the magnetic flux tube $\alpha(s,t)$ along the curve $\alpha$ is visualized for some values of the stretch factor function $r(s)$ in Figure 2.

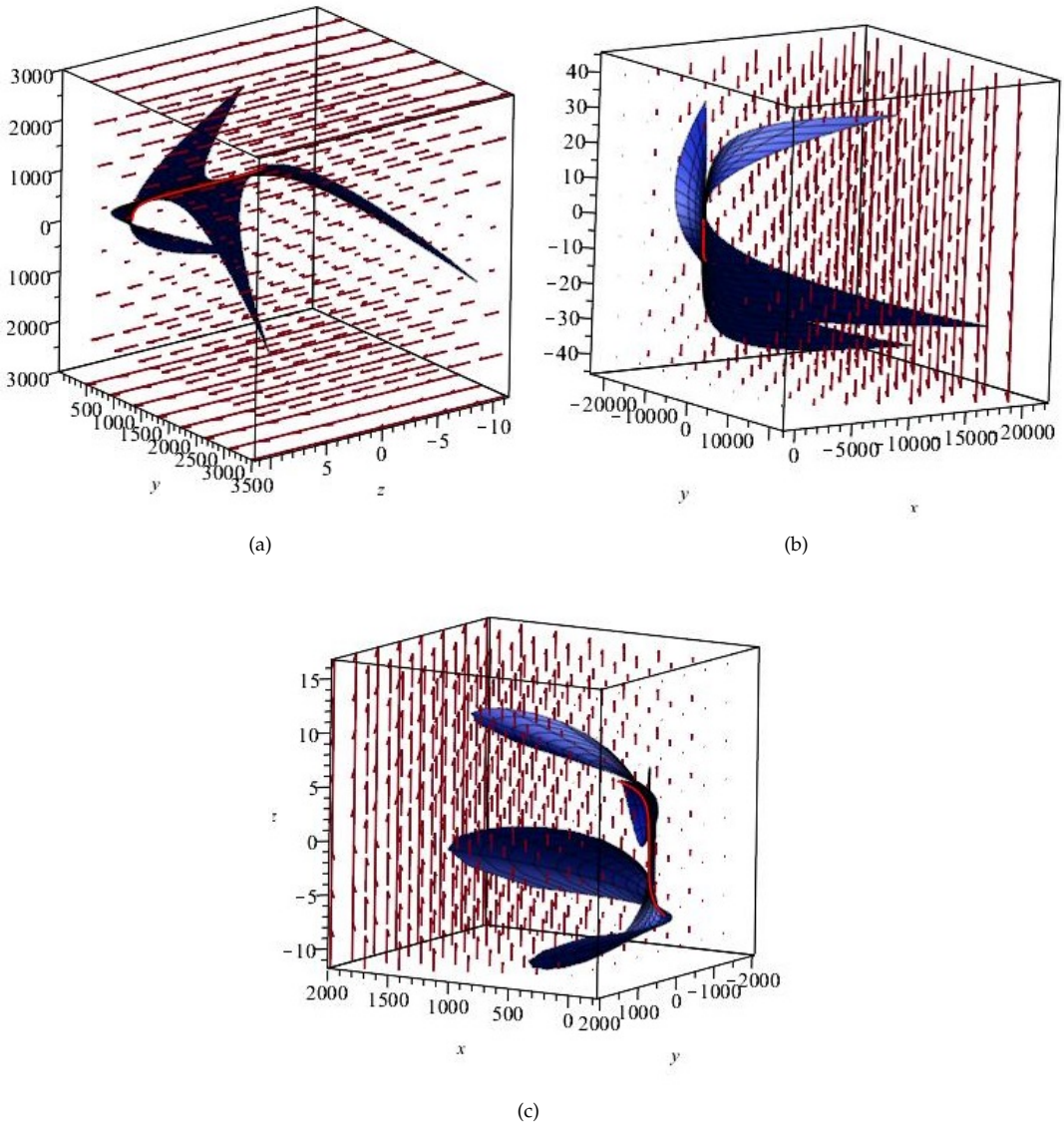


(a) (b)

(c)

**Figure 2.** Flux tube along the neutral points, for the magnetic field $B = (1, -5z, -5y)$, for $r(s) = 1$, $r(s) = 3s$, $r(s) = 3\sin s$, respectively.

**Example 6.2.** Let us take $B = (z, 1, y)$, so we can calculate $\alpha(s)$ via the differential equation $\partial_s \alpha = B$ as $\alpha(s) = (\sinh s, s, \cosh s)$. Thus, the magnetic flux tube along the curve $\alpha$ is visualized in Figure 3.




(a) (b)

(c)

**Figure 3.** Flux tube along the neutral points, for the magnetic field $B = (z, 1, x)$, for $r(s) = 1$, $r(s) = 3s$, $r(s) = 3\sin s$, respectively.

**Example 6.3.** Let us take $B = (1, 5z, 5y)$, then $\alpha(s)$ computed by the differential equation $\partial_s \alpha = B$ as in Example 1. In Figure 4, we show how the magnetic flux tube changes and deforms along this curve as it gets closer to the neutral point. As is known, magnetic field lines are in motion in space, rotate around the neutral point, and are disconnected and reconnected. Therefore, the magnetic flux tube surfaces on which these magnetic field lines lie can also be deformed during this time. We can say more clearly that the purple surface is a magnetic flux tube along a helix curve, and as this flux tube approaches the neutral points, it deforms around the neutral point and turns into a blue magnetic flux tube.

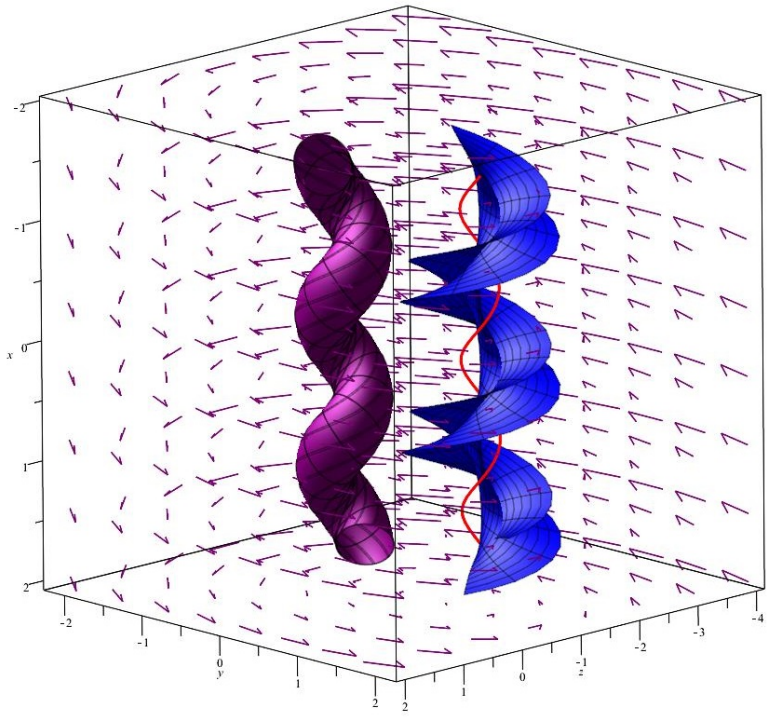


**Figure 4.** the non-deformed magnetic flux tube (purple) and the deformed magnetic flux tube (blue) near the curve $\alpha$ (red) that consist of neutral points for $r(s) = 0.5$.

## 7. Conclusions

One of the key topics in magnetic theory is the motion of light. The forces created as a result of the magnetic field's influence shape this movement. Physicists and geometers are interested in the curves created as a result of these motions, or the trajectory on which it motions. Geometry and kinematics are the two most crucial subjects for investigation if there are motion, trajectory, and curved surfaces at a location. Light moves along a null curve. In this work, we examined the magnetic flux tubes formed along and near the neutral points, which are created as a result of the magnetic field effect and form null curves. Flux tube (magnetic flux tube) quaternions produced along the neutral points are used to obtain the equations. In addition, we researched magnetic field line components of the magnetic flux tube. We are able to do this by looking at the particle's motion in the flux tubes that are built alongside and around the neutral points. We also describe the magnetic field lines that are produced around the neutral points. We are able to determine which curves the magnetic field lines exhibit when the conditions near the neutral points are met owing to the matrix analysis we performed using the magnetic field components. In Section fifth, we demonstrate this section, which yields intriguing geometrical results. Additionally, rotation curves are created when the magnetic field is seen to be tangential to itself. The fan plane is the plane in which these curves spread and converge. Additionally, these rotation curves match the geometric integral curves. Lines of the magnetic field turn and move toward neutral points. It experiences deformation when it reaches these neutral points where the magnetic field is zero. In this work, we used the magnetic field line as the parameter curve and demonstrated how the magnetic flux tube changes and deforms along this curve as it gets closer to the neutral point.
The results of these investigations have enabled us to study the deformation and singularity of neutral points, which produce intriguing geometrical effects, in future research.

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## Competing interests

The authors declare that they have no competing interests.

## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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