

Article

Operators Obtained by Using Certain Generating Function for Approximation

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Abstract: This paper is concerned with the sequence of positive linear operators obtained by certain generating functions of polynomials and with investigation of its approximation properties in detail. Initially, the convergence theorem is expressed for the sequence constructed in this article using the universal Korovkin-type theorem and then considering the modulus of continuity and the Lipschitz class, an estimate of the degree of approximation is obtained for this sequence of positive linear operators. Moreover, the generalization involving integral of these operators is defined and then their approximation properties are examined.

Keywords: Szasz operator; Szasz-Kantorovich operator; Brenke polynomials; Boas-Buck type polynomials; generating functions

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1. Introduction

The most important purpose of approximation theory is to express a given function, for which it is difficult to implement mathematical operations, in terms of functions that are the simplest and most usable mathematically. Algebraic polynomials, trigonometric polynomials and splines can be given as examples of this type of function.

Orthogonal polynomials have a wide range of usage area in mathematical analysis, for example, one of these areas is approximation theory. A sequence of positive linear operators composed by the generating function of Appell polynomials and by the generating function of Sheffer polynomials is studied by Jakimovski-Leviatan [1] and Ismail [2], respectively. It should be noted that the operators defined by Jakimovski-Leviatan and Ismail include operators defined by Szasz [3] for convergence on an unbounded interval.

Quite important positive linear operators sequences obtained by considering the generating function of Appell polynomials, Sheffer polynomials, Brenke polynomials and Boas-Buck polynomials [4,5] are defined and approximation results of these operators are acquired. The articles that can be given as reference for the approximation results obtained for the sequence of operators containing the mentioned polynomials are [6–15].

Suppose that ψ , A , B , C and D functions are analytic on the disc $|z| < \rho$ with $\rho > 1$. Let each of these functions has the following series expansion

$$\begin{aligned}\psi(t) &= \sum_{i=0}^{\infty} \alpha_i t^i, & A(t) &= \sum_{i=0}^{\infty} a_i t^i, & B(t) &= \sum_{i=0}^{\infty} b_i t^{i+1}, \\ C(t) &= \sum_{i=0}^{\infty} c_i t^{i+1}, & D(t) &= \sum_{i=0}^{\infty} d_i t^{i+3},\end{aligned}$$

where the coefficients in the above series $\alpha_i \neq 0$, $a_0 \neq 0$, $c_0 \neq 0$ are satisfied. The polynomials θ_i generated by the relation

$$A(t) \cdot \psi(x^2 B(t) + xC(t) + D(t)) = \sum_{i=0}^{\infty} \theta_i(x) t^i \quad (1)$$

were observed in [16].

This relation allows us to define the following operators

$$\mathcal{W}_n(f; x) = \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) f\left(\frac{i}{n}\right). \quad (2)$$

We restrict ourselves to the following conditions

- (i) $\psi(t) > 0$ for every $t \in \mathbb{R}$,
- (ii) $\theta_i(x) \geq 0$ for every $i = 0, 1, 2, \dots$ and for all $x \geq 0$ and $A(1) > 0$.

Under these conditions, it is easy to see that $\mathcal{W}_n$ operators are positive. In order to obtain the convergence of this sequence of operators using the universal Korovkin-type theorem, we also assume that

$$B'(1) = 0, \quad B''(1) = 0 \quad \text{and} \quad C'(1) = 1$$

are satisfied. Note that if we take $B(t) = D(t) = 0$, it is easily seen that the sequence of operators (2) can be reduced to the one obtained in [7]. Plus, by taking $B(t) = D(t) = 0$ and $\psi(t) = e^t$ in (2), we meet the operators represented in [2]. Finally, Szasz operators [3] can be deduced by choosing $B(t) = D(t) = 0$, $C(t) = t$, $\psi(t) = e^t$ and $A(t) = 1$.

This paper is concerned with the sequence of positive linear operators $(\mathcal{W}_n)_{n \geq 1}$ and with detailed investigation of its approximation properties. In Section 2, firstly, the convergence theorem is expressed for the sequence $(\mathcal{W}_n)_{n \geq 1}$ constructed in this article using the universal Korovkin-type theorem and then considering the modulus of continuity and the Lipschitz class, which are very important concepts in approximation theory, an estimate of the degree of approximation is obtained for this sequence of positive linear operators. In the last section of this article, after defining the following

$$\mathcal{W}_n^*(f; x) = \frac{n}{A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \int_{\frac{i}{n}}^{\frac{i+1}{n}} f(\tau) d\tau \quad (3)$$

Kantorovich generalization of the sequence of operators $(\mathcal{W}_n)_{n \geq 1}$, the approximation results are obtained similarly.

2. Some Results on the Operators $\mathcal{W}_n$

Before we are in a position to apply the Korovkin-type theorem to the sequence of operators $(\mathcal{W}_n)_{n \geq 1}$, we need to have some important lemmas.

Lemma 1. *Under the assumptions $B'(1) = 0$, $B''(1) = 0$ and $C'(1) = 1$, for the positive linear operators $\mathcal{W}_n$,*

$$\begin{aligned} \mathcal{W}_n(1; x) &= 1, \\ \mathcal{W}_n(\xi; x) &= \frac{\psi'(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} x \\ &\quad + \frac{1}{n} \left(\frac{A'(1)}{A(1)} + D'(1) \right) \frac{\psi'(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))}, \end{aligned}$$

$$\begin{aligned}
\mathcal{W}_n(\xi^2; x) &= \frac{\psi''(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))} x^2 \\
&+ \left[\left(\frac{2A'(1)}{A(1)} + C''(1) + 1 \right) \frac{\psi'(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))} \right. \\
&+ 2D'(1) \frac{\psi''(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))} \left. \right] \frac{x}{n} \\
&+ \left[\frac{A''(1) + A'(1)}{A(1)} \right. \\
&+ \left(\frac{2A'(1)D'(1)}{A(1)} + D''(1) + D'(1) \right) \frac{\psi'(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))} \\
&+ (D'(1))^2 \frac{\psi''(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))} \left. \right] \frac{1}{n^2}.
\end{aligned}$$

Proof. If we put $t = 1$ and $x \rightarrow nx$ in the generating function, it follows from (1) that

$$A(1) \cdot \psi(n^2x^2B(1) + nxC(1) + D(1)) = \sum_{i=0}^{\infty} \theta_i(nx),$$

this implies

$$\mathcal{W}_n(1; x) = 1$$

by the definition of operator. Differentiating once with respect to variable t of (1) gives

$$\begin{aligned}
\sum_{i=0}^{\infty} \theta_i(x) i t^{i-1} &= A'(t) \cdot \psi(x^2B(t) + xC(t) + D(t)) \\
&+ A(t) \cdot (x^2B'(t) + xC'(t) + D'(t)) \cdot \psi'(x^2B(t) + xC(t) + D(t)).
\end{aligned} \quad (4)$$

Using the same approach, one can deduce

$$\begin{aligned}
\sum_{i=0}^{\infty} \theta_i(nx) i &= A'(1) \cdot \psi(n^2x^2B(1) + nxC(1) + D(1)) \\
&+ A(1) \cdot (n^2x^2B'(1) + nxC'(1) + D'(1)) \cdot \psi'(n^2x^2B(1) + nxC(1) + D(1)).
\end{aligned}$$

Since by hypothesis $B'(1) = 0$ and $C'(1) = 1$, the above equality gives that

$$\begin{aligned}
\mathcal{W}_n(\xi; x) &= \frac{1}{A(1) \cdot \psi(n^2x^2B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \frac{i}{n} \\
&= \frac{\psi'(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))} x \\
&+ \frac{1}{n} \left(\frac{A'(1)}{A(1)} + D'(1) \right) \frac{\psi'(n^2x^2B(1) + nxC(1) + D(1))}{\psi(n^2x^2B(1) + nxC(1) + D(1))}.
\end{aligned}$$

On the basis of (4), we find that

$$\begin{aligned}
\sum_{i=0}^{\infty} \theta_i(x) i(i-1) t^{i-2} &= A''(t) \cdot \psi(x^2 B(t) + xC(t) + D(t)) \\
&\quad + 2A'(t) \cdot (x^2 B'(t) + xC'(t) + D'(t)) \cdot \psi'(x^2 B(t) + xC(t) + D(t)) \\
&\quad + A(t) \cdot (x^2 B''(t) + xC''(t) + D''(t)) \cdot \psi'(x^2 B(t) + xC(t) + D(t)) \\
&\quad + A(t) \cdot (x^2 B'(t) + xC'(t) + D'(t))^2 \cdot \psi''(x^2 B(t) + xC(t) + D(t)).
\end{aligned}$$

Now using the similar idea, we obtain

$$\begin{aligned}
\sum_{i=0}^{\infty} \theta_i(nx) i^2 &= A''(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1)) \\
&\quad + 2A'(1) \cdot (n^2 x^2 B'(1) + nx C'(1) + D'(1)) \cdot \psi'(n^2 x^2 B(1) + nx C(1) + D(1)) \\
&\quad + A(1) \cdot (n^2 x^2 B''(1) + nx C''(1) + D''(1)) \cdot \psi'(n^2 x^2 B(1) + nx C(1) + D(1)) \\
&\quad + A(1) \cdot (n^2 x^2 B'(1) + nx C'(1) + D'(1))^2 \cdot \psi''(n^2 x^2 B(1) + nx C(1) + D(1)) \\
&\quad + A'(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1)) \\
&\quad + A(1) \cdot (n^2 x^2 B'(1) + nx C'(1) + D'(1)) \cdot \psi'(n^2 x^2 B(1) + nx C(1) + D(1)).
\end{aligned}$$

Then, using the definition of the operator, one can easily obtain

$$\begin{aligned}
\mathcal{W}_n(\xi^2; x) &= \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left(\frac{i}{n}\right)^2 \\
&= \frac{\psi''(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} x^2 \\
&\quad + \left[\left(\frac{2A'(1)}{A(1)} + C''(1) + 1 \right) \frac{\psi'(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} \right. \\
&\quad \left. + 2D'(1) \frac{\psi''(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} \right] \frac{x}{n} \\
&\quad + \left[\frac{A''(1) + A'(1)}{A(1)} \right. \\
&\quad \left. + \left(\frac{2A'(1)D'(1)}{A(1)} + D''(1) + D'(1) \right) \frac{\psi'(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} \right. \\
&\quad \left. + (D'(1))^2 \frac{\psi''(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} \right] \frac{1}{n^2},
\end{aligned}$$

where we have used the $B'(1) = 0$, $B''(1) = 0$ and $C'(1) = 1$. $\square$

Lemma 1 enables us to obtain the following results.

Lemma 2.

$$\begin{aligned}
\mathcal{W}_n(\xi - x; x) &= \left(\frac{\psi'(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} - 1 \right) x \\
&\quad + \left(\frac{A'(1)}{A(1)} + D'(1) \right) \frac{\psi'(n^2 x^2 B(1) + nx C(1) + D(1))}{\psi(n^2 x^2 B(1) + nx C(1) + D(1))} \frac{1}{n}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}_n((\xi - x)^2; x) &= \left[\frac{\psi''(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} \right. \\
&\quad \left. - 2 \frac{\psi'(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} + 1 \right] x^2 \\
&\quad + \left[\left(\frac{2A'(1)}{A(1)} + C''(1) + 1 \right) \frac{\psi'(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} \right. \\
&\quad + 2D'(1) \frac{\psi''(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} \\
&\quad \left. - 2 \left(\frac{A'(1)}{A(1)} + D'(1) \right) \frac{\psi'(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} \right] \frac{x}{n} \\
&\quad + \left[\frac{A''(1) + A'(1)}{A(1)} \right. \\
&\quad + \left(\frac{2A'(1)D'(1)}{A(1)} + D''(1) + D'(1) \right) \frac{\psi'(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} \\
&\quad \left. + (D'(1))^2 \frac{\psi''(n^2 x^2 B(1) + nxC(1) + D(1))}{\psi(n^2 x^2 B(1) + nxC(1) + D(1))} \right] \frac{1}{n^2}.
\end{aligned}$$

Proof. To prove these, we use the following expressions

$$\begin{aligned}
\mathcal{W}_n(\xi - x; x) &= \mathcal{W}_n(\xi; x) - x\mathcal{W}_n(1; x), \\
\mathcal{W}_n((\xi - x)^2; x) &= \mathcal{W}_n(\xi^2; x) - 2x\mathcal{W}_n(\xi; x) + x^2\mathcal{W}_n(1; x),
\end{aligned}$$

which give us desired results by using Lemma 1. $\square$

Now, we shall assume that

$$\lim_{y \rightarrow \pm\infty} \frac{\psi''(y)}{\psi(y)} = 1 \text{ and } \lim_{y \rightarrow \pm\infty} \frac{\psi'(y)}{\psi(y)} = 1$$

and consider these assumptions to investigate convergence of the sequence $(\mathcal{W}_n)_{n \geq 1}$. In the theorem which follows, we are going to use set $\mathcal{B}$ defined as

$$\mathcal{B} = \left\{ f : \lim_{\xi \rightarrow \infty} \frac{f(\xi)}{1 + \xi^2} \in \mathbb{R} \right\}.$$

Theorem 1. Let S be any compact subset of $[0, \infty)$. If f is a function which is continuous on the interval $[0, \infty)$ and belongs to the set $\mathcal{B}$, then the sequence $(\mathcal{W}_n)_{n \geq 1}$ is uniformly convergent to the function f on S .

Proof. For $j = 0, 1, 2$, we can easily calculate that the sequence $(\mathcal{W}_n(\xi^j; x))_{n \geq 1}$ converges uniformly on S to the function $\lambda_j(x) = x^j$. As a consequence of universal Korovkin type theorem [17], uniform convergence mentioned above implies the validity of the assertion of theorem. $\square$

Theorem 2. If

$$\omega(f; \delta) := \sup_{\substack{x, y \in [0, \infty) \\ |x - y| \leq \delta}} |f(x) - f(y)|$$

is the modulus of continuity [18] of the function f which is uniform continuous on $[0, \infty)$, then

$$|\mathcal{W}_n(f; x) - f(x)| \leq 2\omega\left(f; \sqrt{\mathcal{W}_n((\xi - x)^2; x)}\right).$$

Proof. The following inequality follows from the property of the function ω and Lemma 1

$$\begin{aligned} |\mathcal{W}_n(f; x) - f(x)| &= \left| \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left(f\left(\frac{i}{n}\right) - f(x) \right) \right| \\ &\leq \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left| f\left(\frac{i}{n}\right) - f(x) \right| \\ &\leq \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left(1 + \frac{1}{\delta} \left| \frac{i}{n} - x \right| \right) \omega(f; \delta) \\ &= \left(1 + \frac{1}{\delta} \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left| \frac{i}{n} - x \right| \right) \omega(f; \delta). \end{aligned}$$

If we use the Cauchy–Schwarz inequality to estimate the last part of this inequality, then by Lemma 2 we obtain

$$\begin{aligned} \sum_{i=0}^{\infty} \theta_i(nx) \left| \frac{i}{n} - x \right| &= \sum_{i=0}^{\infty} \sqrt{\theta_i(nx)} \sqrt{\theta_i(nx)} \left| \frac{i}{n} - x \right| \\ &\leq \left[\sum_{i=0}^{\infty} \theta_i(nx) \right]^{1/2} \left[\sum_{i=0}^{\infty} \theta_i(nx) \left(\frac{i}{n} - x \right)^2 \right]^{1/2} \\ &= A(1) \cdot \psi(n^2 x^2 B(1) + nxC(1) + D(1)) \cdot \left[\mathcal{W}_n((\xi - x)^2; x) \right]^{1/2}, \end{aligned}$$

which implies that

$$|\mathcal{W}_n(f; x) - f(x)| \leq \left(1 + \frac{1}{\delta} \left[\mathcal{W}_n((\xi - x)^2; x) \right]^{1/2} \right) \omega(f; \delta).$$

The desired result follows by choosing

$$\delta = \left[\mathcal{W}_n((\xi - x)^2; x) \right]^{1/2}.$$

□

For $0 < \beta \leq 1$ and $\eta_1, \eta_2 \in [0, \infty)$, we shall assume that the function ϑ satisfies the following condition

$$|\vartheta(\eta_1) - \vartheta(\eta_2)| \leq M \cdot |\eta_1 - \eta_2|^\beta. \quad (5)$$

Theorem 3. If the function ϑ satisfies the condition given in (5), then

$$|\mathcal{W}_n(\vartheta; x) - \vartheta(x)| \leq M \cdot \left[\mathcal{W}_n((\xi - x)^2; x) \right]^{\frac{\beta}{2}}.$$

Proof. Lemma 1, the positivity of linear operators $\mathcal{W}_n$ and the condition on the function ϑ given in (5) together imply that

$$\begin{aligned} |\mathcal{W}_n(\vartheta; x) - \vartheta(x)| &= |\mathcal{W}_n(\vartheta(\xi) - \vartheta(x); x)| \\ &\leq \mathcal{W}_n(|\vartheta(\xi) - \vartheta(x)|; x) \\ &\leq M \cdot \mathcal{W}_n(|\xi - x|^\beta; x). \end{aligned}$$

By an application of inequality of Hölder, we obtain

$$\begin{aligned}
\mathcal{W}_n(|\xi - x|^\beta; x) &= \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left| \frac{i}{n} - x \right|^\beta \\
&= \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} [\theta_i(nx)]^{\frac{2-\beta}{2}} [\theta_i(nx)]^{\frac{\beta}{2}} \left| \frac{i}{n} - x \right|^\beta \\
&\leq \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \\
&\quad \times \left[A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1)) \right]^{\frac{2-\beta}{2}} \\
&\quad \times \left\{ \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \right\}^{\frac{2-\beta}{2}} \\
&\quad \times \left[A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1)) \right]^{\frac{\beta}{2}} \\
&\quad \times \left\{ \frac{1}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \left| \frac{i}{n} - x \right|^2 \right\}^{\frac{\beta}{2}} \\
&= [\mathcal{W}_n(1; x)]^{\frac{2-\beta}{2}} \left[\mathcal{W}_n((\xi - x)^2; x) \right]^{\frac{\beta}{2}}.
\end{aligned}$$

So, the proof is complete. $\square$

3. An Extension of $\mathcal{W}_n$ Operators

Definitions of the operators $\mathcal{W}_n$ and $\mathcal{W}_n^*$ lead to the following lemma.

Lemma 3. Let $(\mathcal{W}_n^*)_{n \geq 1}$ be a sequence of operators defined as (3). Then,

$$\begin{aligned}
\mathcal{W}_n^*(1; x) &= 1, \\
\mathcal{W}_n^*(\xi; x) &= \mathcal{W}_n(\xi; x) + \frac{1}{2n}, \\
\mathcal{W}_n^*(\xi^2; x) &= \mathcal{W}_n(\xi^2; x) + \frac{1}{n} \mathcal{W}_n(\xi; x) + \frac{1}{3n^2}.
\end{aligned}$$

This result brings us to the following remark.

Remark 1. Reasoning as in the proof of Lemma 2, we obtain

$$\begin{aligned}
\mathcal{W}_n^*(\xi - x; x) &= \mathcal{W}_n(\xi - x; x) + \frac{1}{2n}, \\
\mathcal{W}_n^*((\xi - x)^2; x) &= \mathcal{W}_n((\xi - x)^2; x) + \frac{1}{n} \mathcal{W}_n(\xi - x; x) + \frac{1}{3n^2}.
\end{aligned}$$

Theorem 4. Let f be a continuous function on $[0, \infty)$ and belong to the set $\mathcal{B}$. Then the convergence, as $n \rightarrow \infty$,

$$\mathcal{W}_n^*(f; \cdot) \rightarrow f$$

is uniform on S which is any compact subset of $[0, \infty)$.

Proof. By using a similar argument in Theorem 1, considering Lemma 3, the following convergence

$$\lim_{n \rightarrow \infty} \mathcal{W}_n^*(\xi^j; x) = x^j, j = 0, 1, 2,$$

which is uniform on S , can be obtained. By reason of the theorem on positive linear operators known as universal Korovkin-type theorem [17], we obtain the uniform convergence of the sequence $(\mathcal{W}_n^*(f; \cdot))_{n \geq 1}$ to f on S . $\square$

Theorem 5. If the function f is uniform continuous on $[0, \infty)$, then

$$|\mathcal{W}_n^*(f; x) - f(x)| \leq 2\omega\left(f; \sqrt{\mathcal{W}_n^*((\xi - x)^2; x)}\right).$$

Proof. If similar considerations used in Theorem 2 apply to the operators $\mathcal{W}_n^*$, then we have

$$\begin{aligned} |\mathcal{W}_n^*(f; x) - f(x)| &\leq \frac{n}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \int_{\frac{i}{n}}^{\frac{i+1}{n}} |f(\tau) - f(x)| d\tau \\ &\leq \frac{n}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \\ &\quad \times \sum_{i=0}^{\infty} \theta_i(nx) \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(1 + \frac{1}{\delta} |\tau - x|\right) \omega(f; \delta) d\tau \\ &= \left(1 + \frac{1}{\delta} \frac{n}{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sum_{i=0}^{\infty} \theta_i(nx) \int_{\frac{i}{n}}^{\frac{i+1}{n}} |\tau - x| d\tau\right) \\ &\quad \times \omega(f; \delta). \end{aligned}$$

Considering the infinite sum on the right hand side of the above expression, the Schwarz inequality implies

$$\begin{aligned} \sum_{i=0}^{\infty} \theta_i(nx) \int_{\frac{i}{n}}^{\frac{i+1}{n}} |\tau - x| d\tau &\leq \frac{1}{\sqrt{n}} \sum_{i=0}^{\infty} \theta_i(nx) \left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} (\tau - x)^2 d\tau \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{n}} \sqrt{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))} \sqrt{\mathcal{W}_n^*(1; x)} \\ &\quad \times \sqrt{\frac{A(1) \cdot \psi(n^2 x^2 B(1) + nx C(1) + D(1))}{n}} \sqrt{\mathcal{W}_n^*((\xi - x)^2; x)}, \end{aligned}$$

from which we obtain

$$|\mathcal{W}_n^*(f; x) - f(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{\mathcal{W}_n^*((\xi - x)^2; x)}\right) \omega(f; \delta).$$

This proves the desired result if we choose

$$\delta = \sqrt{\mathcal{W}_n^*((\xi - x)^2; x)}.$$

□

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