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A Generalization of Szasz Operators by Using the Appell Polynomials of Class $A^{(2)}$

Serhan Varma * and Sezgin Sucu

Department of Mathematics, Faculty of Science, Ankara University, Ankara TR-06100, Turkey; ssucu@ankara.edu.tr
* Correspondence: svarma@science.ankara.edu.tr

Abstract: In this paper, as a generalization of Szasz operators, a brand-new sequence of operators including the Appell polynomials of class $A^{(2)}$ is introduced. First, the convergence of this new sequence of operators is obtained, and then, some approximation results are presented by using the tools of approximation theory. In addition, an explicit example for this kind of sequence of operators containing Gould–Hopper polynomials is introduced. The error of the approximation of this new sequence of operators to a function is established.

Keywords: Szasz operator; Appell polynomials of class $A^{(2)}$; modulus of continuity; Steklov function

MSC: 41A10; 41A36; 33C45

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1. Introduction

Szasz operators [1] are extensions of Bernstein operators to infinite intervals, and these operators play a very important role in the field of approximation theory. The generalizations of Szasz operators by using polynomials, especially defined via generating functions, have been frequently studied lately. These kinds of generalizations provide a range of new sequences of operators to approximation theory. Jakimovski and Leviatan [2] presented a generalization of Szasz operators via Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function in the disc $|z| < R$ $(R > 1)$ and $g(1) \neq 0$. The Appell polynomials $p_k(x)$ have generating functions of the following form:

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k .$$

Under the restriction $p_k(x) \geq 0$ for $x \in [0, \infty)$, Jakimovski and Leviatan constructed the linear positive operators $P_n(f;x)$ by

$$P_n(f;x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \quad , \quad \text{for } n \in \mathbb{N}$$

and obtained the approximation properties of this sequence of operators. Then, Ismail [3] defined another generalization of Szasz operators and also Jakimovski and Leviatan operators through the instrument of Sheffer polynomials. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 \neq 0)$ and $H(z) = \sum_{k=1}^{\infty} h_k z^k (h_1 \neq 0)$ be analytic functions in the disc $|z| < R$ $(R > 1)$, where a_k and h_k are real. The Sheffer polynomials $p_k(x)$ have generating functions of the type

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x)t^k, \qquad |t| < R .$$

With the help of following restrictions

$$(i) \text{ for } x \in [0,\infty),\ p_k(x) \geq 0,$$
$$(ii)\ A(1) \neq 0 \text{ and } H'(1) = 1,$$

Ismail investigated the convergence properties of linear positive operators given by

$$\mathcal{L}_n(f;x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \ , \quad \text{for } n \in \mathbb{N}\,.$$

One can find more generalizations of Szasz operators using similar methods in the literature [4–12].

In this contribution, we introduce a brand-new generalization of Szasz operators with the help of the Appell polynomials of class $A^{(2)}$ defined by Kazmin [13]. The Appell polynomials $p_k(x)$ of class $A^{(2)}$ are given by the following generating function:

$$A(t)e^{xt} + B(t)e^{-xt} = \sum_{k=0}^{\infty} p_k(x)t^k \tag{1}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \ \text{ and } \ B(t) = \sum_{k=0}^{\infty} \frac{b_k}{k!}t^k$$

are formal power series defined at the disc $|z| < R$ $(R > 1)$ with $a_0^2 - b_0^2 \neq 0$. Hermite polynomials, Bernoulli polynomials and Euler polynomials are examples of these types of polynomials. By using Appell polynomials of class $A^{(2)}$ given by (1), we define the sequence of operators for $x \in [0,\infty)$

$$T_n(f;x) = \frac{1}{A(1)e^{nx} + B(1)e^{-nx}} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{2}$$

with the restrictions $A(1) > 0$, $B(1) \geq 0$ and $p_k(x) > 0$ for all $k = 0, 1, \ldots$. These restrictions assure us of the positivity of the sequence of operators in (2). Note that for the special case $A(t) = 1$ and $B(t) = 0$, we discover the well-known Szasz operators again.

Some numerical examples involving special kinds of orthogonal polynomials such as Gould–Hopper polynomials can be constructed by using the sequence of operators in (2). Moreover, one can derive other sequences of operators by choosing $A(t)$ and $B(t)$ from (1) in view of the restrictions $A(1) > 0$, $B(1) \geq 0$ and $p_k(x) > 0$ for all $k = 0, 1, \ldots$. These operators can be used in applications of computer simulations, in data science, in speech analysis problems and also in image processing. In this paper, first, the convergence properties of the sequence of operators in (2) are studied. Then, some estimations for approximation results are obtained by using the modulus of continuity, Lipschitz class functions and the second-order modulus of continuity. Finally, a numerical example is presented.

2. Convergence of the Operators and Some Approximation Results

Let us first obtain some equalities. We will use these equalities further.

Lemma 1.

$$\begin{aligned} T_n(1;x) &= 1, \\ T_n(\xi;x) &= \left[\frac{A(1)e^{nx}-B(1)e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right]x+\frac{A^{'}(1)e^{nx}+B^{'}(1)e^{-nx}}{n[A(1)e^{nx}+B(1)e^{-nx}]}, \\ T_n\left(\xi^2;x\right) &= x^2+\frac{x}{n}\left\{\frac{\left[2A^{'}(1)+A(1)\right]e^{nx}-\left[2B^{'}(1)+B(1)\right]e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right\} \\ &\quad +\frac{\left[A^{''}(1)+A^{'}(1)\right]e^{nx}+\left[B^{''}(1)+B^{'}(1)\right]e^{-nx}}{n^2[A(1)e^{nx}+B(1)e^{-nx}]}. \end{aligned}$$

Proof. By taking $t=1$ and nx instead of x in (1), we obtain

$$A(1)e^{nx}+B(1)e^{-nx}=\sum_{k=0}^{\infty}p_k(nx).$$

This provides us with the following:

$$T_n(1;x)=\frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)=1.$$

First taking the derivative of both sides of (1) with respect to t and then using above approach, $t=1$ and nx instead of x, leads to

$$\sum_{k=0}^{\infty}kp_k(nx)=nx\left[A(1)e^{nx}-B(1)e^{-nx}\right]+A^{'}(1)e^{nx}+B^{'}(1)e^{-nx}. \tag{3}$$

Thus, we have

$$\begin{aligned} T_n(\xi;x) &= \frac{1}{n[A(1)e^{nx}+B(1)e^{-nx}]}\sum_{k=0}^{\infty}kp_k(nx) \\ &= \left[\frac{A(1)e^{nx}-B(1)e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right]x+\frac{A^{'}(1)e^{nx}+B^{'}(1)e^{-nx}}{n[A(1)e^{nx}+B(1)e^{-nx}]}. \end{aligned}$$

Applying the second derivative to (1) with respect to t gives

$$\begin{aligned} \sum_{k=0}^{\infty}\left(k^2-k\right)p_k(x)t^{k-2} &= \left[A^{''}(t)+xA^{'}(t)\right]e^{xt}+\left[A^{'}(t)+xA(t)\right]xe^{xt} \\ &\quad +\left[B^{''}(t)-xB^{'}(t)\right]e^{-xt}+\left[B^{'}(t)-xB(t)\right](-x)e^{-xt}. \end{aligned}$$

We conclude the following equality in view of $t=1$, nx instead of x and (3):

$$\begin{aligned} \sum_{k=0}^{\infty}k^2p_k(nx) &= n^2x^2\left[A(1)e^{nx}+B(1)e^{-nx}\right]+nx\left\{\left[2A^{'}(1)+A(1)\right]e^{nx}\right. \\ &\quad \left.-\left[2B^{'}(1)+B(1)\right]e^{-nx}\right\}+\left[A^{''}(1)+A^{'}(1)\right]e^{nx} \\ &\quad +\left[B^{''}(1)+B^{'}(1)\right]e^{-nx}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} T_n\left(\xi^2;x\right) &= \frac{1}{n^2[A(1)e^{nx}+B(1)e^{-nx}]}\sum_{k=0}^{\infty}k^2p_k(nx) \\ &= x^2+\frac{x}{n}\left\{\frac{\left[2A^{'}(1)+A(1)\right]e^{nx}-\left[2B^{'}(1)+B(1)\right]e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right\} \\ &\quad +\frac{\left[A^{''}(1)+A^{'}(1)\right]e^{nx}+\left[B^{''}(1)+B^{'}(1)\right]e^{-nx}}{n^2[A(1)e^{nx}+B(1)e^{-nx}]}. \end{aligned}$$

□

Lemma 2.

$$\begin{aligned} T_n(\xi-x;x) &= -\frac{2B(1)e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}x+\frac{A^{'}(1)e^{nx}+B^{'}(1)e^{-nx}}{n[A(1)e^{nx}+B(1)e^{-nx}]}, \\ T_n\left((\xi-x)^2;x\right) &= x^2\left[\frac{4B(1)e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right] \\ &\quad +\frac{x}{n}\left[\frac{A(1)e^{nx}-\left[4B^{'}(1)+B(1)\right]e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right] \\ &\quad +\frac{\left[A^{''}(1)+A^{'}(1)\right]e^{nx}+\left[B^{''}(1)+B^{'}(1)\right]e^{-nx}}{n^2[A(1)e^{nx}+B(1)e^{-nx}]}. \end{aligned}$$

Proof. The above identities can easily be found from the following equalities:

$$T_n(\xi-x;x)=T_n(\xi;x)-xT_n(1;x),$$

$$T_n\left((\xi-x)^2;x\right)=T_n\left(\xi^2;x\right)-2xT_n(\xi;x)+x^2T_n(1;x).$$

□

Now, we are able to prove our main theorem.

Theorem 1. *Let f be continuous on $[0,\infty)$ and belong to the class*

$$E=\left\{f:\frac{f(x)}{1+x^2}\ \textit{is convergent when}\ x\to\infty\right\}.$$

Then, the sequence of operators in (2) converges uniformly on the compact subsets of the interval $[0,\infty)$, i.e.,

$$\lim_{n\to\infty}T_n(f;x)=f(x).$$

Proof. By using Lemma 1, we obtain

$$\lim_{n\to\infty}T_n\left(\xi^i;x\right)=x^i,\ i=0,1,2.$$

These convergences are satisfied uniformly on the compact subsets of the interval $[0,\infty)$. Hence, the proof is provided by the universal Korovkin theorem [14]. □

From now on, we represent the approximation results.

Theorem 2.

$$|T_n(f;x)-f(x)| \leq 2\omega\left(f;\sqrt{T_n\left((\xi-x)^2;x\right)}\right),$$

where ω is the modulus of continuity of the function f [15] defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\leq\delta}} |f(x)-f(y)|$$

and f is uniform continuous on the interval $[0,\infty)$.

Proof. Using the well-known property of $\omega(f;\delta)$ and the triangle inequality gives us

$$\begin{aligned} |T_n(f;x)-f(x)| &= \left|\frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\left[f\left(\frac{k}{n}\right)-f(x)\right]\right| \\ &\leq \frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\leq \frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\left[1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right]\omega(f;\delta) \\ &= \left\{1+\frac{1}{\delta}\frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\left|\frac{k}{n}-x\right|\right\}\omega(f;\delta). \end{aligned} \tag{4}$$

Considering the Cauchy–Schwarz inequality leads us to

$$\begin{aligned} \sum_{k=0}^{\infty}p_k(nx)\left|\frac{k}{n}-x\right| &= \sum_{k=0}^{\infty}\sqrt{p_k(nx)}\sqrt{p_k(nx)}\left|\frac{k}{n}-x\right| \\ &\leq \left[\sum_{k=0}^{\infty}p_k(nx)\right]^{1/2}\left[\sum_{k=0}^{\infty}p_k(nx)\left(\frac{k}{n}-x\right)^2\right]^{1/2} \\ &= \left[A(1)e^{nx}+B(1)e^{-nx}\right]\left[T_n\left((\xi-x)^2;x\right)\right]^{1/2}. \end{aligned}$$

If we substitute the last inequality into (4), we have

$$|T_n(f;x)-f(x)| \leq \left\{1+\frac{1}{\delta}\left[T_n\left((\xi-x)^2;x\right)\right]^{1/2}\right\}\omega(f;\delta).$$

Here, by choosing

$$\delta = \left[T_n\left((\xi-x)^2;x\right)\right]^{1/2},$$

we obtain the desired result. □

Now, for $0 < \gamma \leq 1$ and $\xi_1, \xi_2 \in [0,\infty)$, let us introduce the following class of functions:

$$Lip_M^{(\gamma)} = \left\{\varphi : |\varphi(\xi_1)-\varphi(\xi_2)| \leq M|\xi_1-\xi_2|^{\gamma}\right\}.$$

Theorem 3. *Assume that $\varphi \in Lip_M^{(\gamma)}$. Then,*

$$|T_n(\varphi;x)-\varphi(x)| \leq M\left[T_n\left((\xi-x)^2;x\right)\right]^{\frac{\gamma}{2}}.$$

Proof. Since $\varphi \in Lip_M^{(\gamma)}$, we obtain

$$\begin{aligned} |T_n(\varphi;x)-\varphi(x)| &= |T_n(\varphi(\xi)-\varphi(x);x)| \\ &\leq T_n(|\varphi(\xi)-\varphi(x)|;x) \\ &\leq MT_n\left(|\xi-x|^{\gamma};x\right). \end{aligned} \tag{5}$$

If we use the Hölder inequality at the right-hand side of the inequality in (5), we obtain

$$\begin{aligned} T_n\left(|\xi-x|^{\gamma};x\right) &= \frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\left|\frac{k}{n}-x\right|^{\gamma} \\ &= \frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}[p_k(nx)]^{\frac{2-\gamma}{2}}[p_k(nx)]^{\frac{\gamma}{2}}\left|\frac{k}{n}-x\right|^{\gamma} \\ &\leq \frac{1}{A(1)e^{nx}+B(1)e^{-nx}} \\ &\quad \times\left[A(1)e^{nx}+B(1)e^{-nx}\right]^{\frac{2-\gamma}{2}}\left\{\frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\right\}^{\frac{2-\gamma}{2}} \\ &\quad \times\left[A(1)e^{nx}+B(1)e^{-nx}\right]^{\frac{\gamma}{2}}\left\{\frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\sum_{k=0}^{\infty}p_k(nx)\left(\frac{k}{n}-x\right)^2\right\}^{\frac{\gamma}{2}} \\ &= [T_n(1;x)]^{\frac{2-\gamma}{2}}\left[T_n\left((\xi-x)^2;x\right)\right]^{\frac{\gamma}{2}}. \end{aligned}$$

Thus, we prove the desired result. □

Let us first introduce Rasa's result and the second-order Steklov function, which are used in the following theorem.

Let $z \in C^2[0,a]$ and $(L_n)_{n\geq 0}$ be a sequence of linear positive operators with the property $L_n(e_0;x)=e_0(x)$, $e_i(\xi)=\xi^i$, $i\in\{0,1,2\}$. Then, Rasa's result [16] is known as

$$|L_n(z;x)-z(x)| \leq \left\|z'\right\|\sqrt{L_n\left((\xi-x)^2;x\right)}+\frac{1}{2}\left\|z''\right\|L_n\left((\xi-x)^2;x\right).$$

For $f\in C[a,b]$, the second-order Steklov function of f [17] is defined by

$$f_h(x) := \frac{1}{h}\int_{-h}^{h}\left(1-\frac{|t|}{h}\right)f(h;x+t)dt, \quad x\in[a,b],$$

where $f(h;.):[a-h,b+h]\to\mathbb{R}$, $h>0$ by

$$f(h;x)=\begin{cases} P_-(x) & ; \quad a-h\leq x\leq a \\ f(x) & ; \quad a\leq x\leq b \\ P_+(x) & ; \quad b<x\leq b+h \end{cases}$$

and P_-, P_+ are the linear best approximations to f on the indicated intervals.

Theorem 4. *Suppose that φ is a continuous function on $[0,\infty)$. Then, we have*

$$|T_n(\varphi;x)-\varphi(x)| \leq \frac{3}{2}\left(1+\frac{a}{2}+\frac{h^2}{2}\right)\omega_2(\varphi;h)+\frac{2h^2}{a}\|\varphi\|.$$

Here, ω_2 is the second-order modulus of continuity of the function φ [15] defined by

$$\omega_2(\varphi;\delta) := \sup_{0<t\leq\delta}\|\varphi(.+2t)-2\varphi(.+t)+\varphi(.)\|.$$

Proof. By using some simple computations, it becomes

$$\begin{aligned} |T_n(\varphi;x)-\varphi(x)| &\leq T_n(|\varphi-\varphi_h|;x)+|T_n(\varphi_h;x)-\varphi_h(x)|+|\varphi_h(x)-\varphi(x)| \\ &\leq 2\|\varphi-\varphi_h\|+|T_n(\varphi_h;x)-\varphi_h(x)|, \end{aligned} \tag{6}$$

where φ_h is the second-order Steklov function of φ. In view of the fact that $\varphi_h \in C^2[0,a]$, Rasa's result given above and the Landau inequality, we derive

$$\begin{aligned} |T_n(\varphi_h;x)-\varphi_h(x)| &\leq \left\|\varphi_h'\right\|\sqrt{T_n\left((\xi-x)^2;x\right)}+\frac{1}{2}\left\|\varphi_h''\right\|T_n\left((\xi-x)^2;x\right) \\ &\leq \left(\frac{2}{a}\|\varphi_h\|+\frac{a}{2}\left\|\varphi_h''\right\|\right)\sqrt{T_n\left((\xi-x)^2;x\right)} \\ &\quad +\frac{1}{2}\left\|\varphi_h''\right\|T_n\left((\xi-x)^2;x\right) \\ &\leq \left(\frac{2}{a}\|\varphi\|+\frac{3a}{4h^2}\omega_2(\varphi;h)\right)\sqrt{T_n\left((\xi-x)^2;x\right)} \\ &\quad +\frac{3}{4h^2}\omega_2(\varphi;h)T_n\left((\xi-x)^2;x\right). \end{aligned} \tag{7}$$

The following relation between the second-order Steklov function and $\omega_2(\varphi;h)$ was given by Zhuk [17]

$$\|\varphi-\varphi_h\| \leq \frac{3}{4}\omega_2(\varphi;h).$$

If we substitute this inequality and (7) into (6), we obtain

$$\begin{aligned} |T_n(\varphi;x)-\varphi(x)| &\leq \frac{3}{2}\omega_2(\varphi;h)+\left(\frac{2}{a}\|\varphi\|+\frac{3a}{4h^2}\omega_2(\varphi;h)\right)\sqrt{T_n\left((\xi-x)^2;x\right)} \\ &\quad +\frac{3}{4h^2}\omega_2(\varphi;h)T_n\left((\xi-x)^2;x\right). \end{aligned}$$

Thus, we obtain the desired result by choosing $h=\sqrt[4]{T_n\left((\xi-x)^2;x\right)}$. □

3. Numerical Example

Gould–Hopper polynomials [18] have generating functions of the form

$$e^{ht^{d+1}}\exp(xt)=\sum_{k=0}^{\infty} g_k^{d+1}(x,h)\frac{t^k}{k!} \tag{8}$$

and the explicit representations can be given by

$$g_k^{d+1}(x,h)=\sum_{s=0}^{\left[\frac{k}{d+1}\right]}\frac{k!}{s!(k-(d+1)s)!}h^s x^{k-(d+1)s}\ .$$

Gould–Hopper polynomials $g_k^{d+1}(x,h)$ are d-orthogonal polynomial sets of Hermite type [19]. Van Iseghem [20] and Maroni [21] discovered the notion of d-orthogonality. Gould–Hopper polynomials are the Appell polynomials of class $A^{(2)}$ by choosing

$$A(t)=e^{ht^{d+1}}\ ,\quad B(t)=0\ .$$

Under the assumption $h \geq 0$, the restrictions $A(1) > 0$, $B(1) \geq 0$ and $p_k(x) > 0$ for all $k = 0, 1, \ldots$ are satisfied. With the help of the generating functions in (8), we obtain the explicit form of the sequence of operators involving Gould–Hopper polynomials $\mathcal{T}_n^*$ by

$$\mathcal{T}_n^*(f;x) = e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} f\left(\frac{k}{n}\right)$$

where $x \in [0, \infty)$. The error of the approximation of the function $f(x) = \frac{x^2}{\sqrt{1+x^2}}$ by using the sequence of operators involving Gould–Hopper polynomials $\mathcal{T}_n^*$ is presented in Table 1. Each of the estimates depending on the parameter h and the special case $d = 3$ is listed in the following table as follows:

Table 1. The error estimation of function f by using modulus of continuity.

n	Estimation for $h = 0.00001$	Estimation for $h = 1$	Estimation for $h = 1.8$
10	0.6423250428	1.1805569940	1.4096205150
10^2	0.2100772120	0.2409468484	0.2812002468
10^3	0.0668966700	0.0679554484	0.0695336528
10^4	0.0211952592	0.0212291148	0.0212804728
10^5	0.0067064284	0.0067075012	0.0067091310
10^6	0.0021211432	0.0021211772	0.0021212294
10^7	0.0006708028	0.0006708042	0.0006708056

4. Concluding Remarks

In this contribution, we present a generalization of Szasz operators by using the Appell polynomials of class $A^{(2)}$. The convergence properties and approximation results of the sequence of operators in (2) are obtained. In addition, a numerical example is given by using Gould–Hopper polynomials.

In further studies, a new sequence of operators which is a generalization of the sequence of operators in (2) can be investigated. For example, one can construct a Kantorovich generalization of the sequence of operators in (2) for the approximation of integrable functions. In addition, this type of sequence of operators can impact various scientific fields.

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