Mathematical Foundations of Computing
Vol. 5, No. 3, August 2022, pp. 173–185
doi:10.3934/mfc.2021026

ON BIVARIATE JAIN OPERATORS

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Abstract. In this paper we deal with bivariate extension of Jain operators. Using elementary method, we show that these opearators are non-increasing in n when the attached function is convex. Moreover, we demonstrate that these operators preserve the properties of modulus of continuity. Finally, we present a Voronovskaja type theorem for the sequence of bivariate Jain operators.

1. **Introduction.** In [15], Jain constructed the following positive linear operators

$$P_{n,1}^{\beta}(f;x)=\begin{cases}\sum\limits_{k=0}^{\infty}\omega_{\beta}(k,nx)f\left(\frac{k}{n}\right), & x>0,\ n\in\mathbb{N},\\ f(0), & x=0,\end{cases} \tag{1}$$

where

$$\omega_{\beta}(k,nx):=\frac{nx\left(nx+k\beta\right)^{k-1}}{k!}e^{-(nx+k\beta)},\ 0\leq\beta<1,\ x>0, \tag{2}$$

is a Poisson type distribution satisfying

$$\sum_{k=0}^{\infty}\omega_{\beta}(k,nx)=1, \tag{3}$$

and f is any function from $C[0,\infty)$, the space of real valued continuous functions on $[0,\infty)$, such that the series is convergent. In the paper, the author studied convergence and the order of convergence by the sequence $\left\{P_{n,1}^{\beta}(f;x)\right\}_{n\geq1}$ on any finite closed interval of $[0,\infty)$ by taking $\beta=\beta_n$ satisfying $\beta_n\to0$ as $n\to\infty$. Note that in the case of $\beta=0$ these operators reduce to the well known Szász–Mirakjan operators [17], [23].

Umar and Razi [25] defined a Kantorovich type generalization of the Jain operators. Farcaş [11] proved a Voronovskaja type theorem for Jain operators. Tarabie [24] studied A-statistical convergence for the sequence of the Jain-beta operators. Farcaş [12] considered a new generalization of Jain's operators of two variables and studied convergence properties of these operators with the help of A-statistical approximation.

Gupta et al. [13] studied a Durrmeyer type generalization of Jain operators. Agratini [2] studied local approximation results and statistical convergence property of the sequence of operators $P_{n,1}^{\beta}$. Gupta and Greubel [14] presented an interesting

2020 *Mathematics Subject Classification.* Primary: 41A25, 41A36; Secondary: 41A63.
Key words and phrases. Bivariate Jain operator, modulus of continuity, monotonicity, Voronovskaja type theorem.

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observation that the Durrmeyer variant of the Jain operators has an advantage that for the convergence; β does not have to be taken as a convergent sequence to zero. Dhamija and Deo [9] also considered Jain–Durrmeyer operators associated with the inverse Pólya–Eggenberger distribution and study uniform approximation and degree of approximation by the sequence of the new modified operators. Abel and Agratini [1], obtained complete asymptotic expansion for Jain operators. Olgun et al. [19] defined a generalization of Jain operators and proved a Voronovskaja type theorem. Doğru et al. [10] considered a generalization of Jain operators preserving linear functions and obtained an error estimate better than the Jain operators. Özarslan [20] studied Stancu type Jain operators and constructed a modification of them so that the modified operators preserve linear functions and provide a better error estimation than the Jain operators and obtained global results in a certain subspace of $C[0,\infty)$, whereas Jain operators don't satisfy. Deniz [8] provided some quantitative estimates for Kantorovich generalization of Jain operators. For a detailed overview concerning Jain operators, we refer to Agratini's paper [3]. Çetin and Başcanbaz-Tunca [6] extended Jain operators in the Schurer's setting and studied some classical problems concerning the new operators.

In the present paper, we consider bivariate extension of the Jain operators and prove that these operators are non-increasing with respect to n, when the attached function is convex. We show that these operators preserve the properties of the modulus of continuity. Finally, we present a Voronovskaja type result for the sequence of bivariate Jain operators.

Throughout the paper, $D \subseteq \mathbb{R}^2$ denotes the set

$$D := \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x < \infty,\ 0 \leq y < \infty\right\}$$

and $C_B(D)$ denotes the space of all real valued continuous and bounded functions defined on D. For $f((x,y))$, where $(x,y) \in D$; we simply write $f(x,y)$. $\mathbf{1}$ denotes the constant function on D such that $\mathbf{1}(x,y) = 1$ for $(x,y) \in D$ and for any (x_1,y_1), $(x_2,y_2) \in D$, $(x_1,y_1) \leq (x_2,y_2)$ means that $x_1 \leq x_2$ and $y_1 \leq y_2$.

We consider the following bivariate extension of the Jain operators, denoted by P_n^β, $n \in \mathbb{N}$, and defined as

$$P_n^\beta(f;x,y) = \begin{cases} \sum\limits_{k=0}^{\infty}\sum\limits_{l=0}^{\infty} \omega_\beta(k,nx)\omega_\beta(l,ny)\, f\left(\frac{k}{n},\frac{l}{n}\right), & (x,y) \in D\setminus\{(0,0)\},\ n \in \mathbb{N}, \\ f(0,0), & (x,y) = (0,0) \end{cases} \tag{4}$$

where $0 \leq \beta < 1$, ω_β is given by (2) and $f \in C_B(D)$. It is clear that the operators P_n^β are positive and linear, and from (3), one has

$$\begin{aligned} P_n^\beta(\mathbf{1};x,y) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \omega_\beta(k,nx)\,\omega_\beta(l,ny) \\ &= 1. \end{aligned} \tag{5}$$

Let $\pi_i : D \to \mathbb{R}$, $i = 1,2$, denote the ith coordinate function defined as $\pi_1(x,y) := x$, and $\pi_2(x,y) = y$, for $(x,y) \in D$. Now, we present some moments and central moments of the bivariate Jain operators P_n^β, $n \in \mathbb{N}$.

Lemma 1.1. *For the operators P_n^β, $n \in \mathbb{N}$, we have*

$$P_n^\beta(\pi_1;x,y) = \frac{x}{1-\beta},$$

$$
\begin{aligned}
P_n^\beta\left(\pi_2;x,y\right) =& \frac{y}{1-\beta},\\
P_n^\beta\left(\pi_1^2;x,y\right) =& \frac{x^2}{\left(1-\beta\right)^2}+\frac{x}{n\left(1-\beta\right)^3},\\
P_n^\beta\left(\pi_2^2;x,y\right) =& \frac{y^2}{\left(1-\beta\right)^2}+\frac{y}{n\left(1-\beta\right)^3},\\
P_n^\beta\left(\pi_1^3;x,y\right) =& \frac{x^3}{\left(1-\beta\right)^3}+\frac{3x^2}{n\left(1-\beta\right)^4}+\frac{\left(2\beta+1\right)x}{n^2\left(1-\beta\right)^5},\\
P_n^\beta\left(\pi_2^3;x,y\right) =& \frac{y^3}{\left(1-\beta\right)^3}+\frac{3y^2}{n\left(1-\beta\right)^4}+\frac{\left(2\beta+1\right)y}{n^2\left(1-\beta\right)^5},\\
P_n^\beta\left(\pi_1^4;x,y\right) =& \frac{x^4}{\left(1-\beta\right)^4}+\frac{6x^3}{n\left(1-\beta\right)^5}+\frac{\left(8\beta+7\right)x^2}{n^2\left(1-\beta\right)^6}+\frac{\left(6\beta^2+8\beta+1\right)x}{n^3\left(1-\beta\right)^7},\\
P_n^\beta\left(\pi_2^4;x,y\right) =& \frac{y^4}{\left(1-\beta\right)^4}+\frac{6y^3}{n\left(1-\beta\right)^5}+\frac{\left(8\beta+7\right)y^2}{n^2\left(1-\beta\right)^6}+\frac{\left(6\beta^2+8\beta+1\right)y}{n^3\left(1-\beta\right)^7}.
\end{aligned}
$$

Proof. From the definition of the operators (4), we have

$$P_n^\beta\left(\pi_1^v;x,y\right)=P_{n,1}^\beta\left(e_v;x\right) \text{ and } P_n^\beta\left(\pi_2^v;x,y\right)=P_{n,1}^\beta\left(e_v;y\right),$$

where $e_v\left(t\right)=t^v,\ v\in\mathbb{N}$. Thus, the moments can be obtained from the corresponding values of univariate Jain operators $P_{n,1}^\beta$ (see [14] or [19]). □

Lemma 1.2. *For the operators $P_n^\beta,\ n\in\mathbb{N}$, we have*

$$
\begin{aligned}
P_n^\beta\left(\pi_1-\mathbf{1}x;x,y\right) =& \frac{\beta x}{1-\beta},\\
P_n^\beta\left(\pi_2-\mathbf{1}y;x,y\right) =& \frac{\beta y}{1-\beta},\\
P_n^\beta\left(\left(\pi_1-\mathbf{1}x\right)^2;x,y\right) =& \frac{\beta^2x^2}{\left(1-\beta\right)^2}+\frac{x}{n\left(1-\beta\right)^3},\\
P_n^\beta\left(\left(\pi_2-\mathbf{1}y\right)^2;x,y\right) =& \frac{\beta^2y^2}{\left(1-\beta\right)^2}+\frac{y}{n\left(1-\beta\right)^3},\\
P_n^\beta\left(\left(\pi_1-\mathbf{1}x\right)\left(\pi_2-\mathbf{1}y\right);x,y\right) =& \frac{\beta^2xy}{\left(1-\beta\right)^2},\\
P_n^\beta\left(\left(\pi_1-\mathbf{1}x\right)^4;x,y\right) =& \frac{\beta^4x^4}{\left(1-\beta\right)^4}+\frac{6\beta^2x^3}{n\left(1-\beta\right)^5}\\
&+\frac{\left(8\beta^2+4\beta+3\right)x^2}{n^2\left(1-\beta\right)^6}+\frac{\left(6\beta^2+8\beta+1\right)x}{n^3\left(1-\beta\right)^7},\\
P_n^\beta\left(\left(\pi_2-\mathbf{1}y\right)^4;x,y\right) =& \frac{\beta^4y^4}{\left(1-\beta\right)^4}+\frac{6\beta^2y^3}{n\left(1-\beta\right)^5}\\
&+\frac{\left(8\beta^2+4\beta+3\right)y^2}{n^2\left(1-\beta\right)^6}+\frac{\left(6\beta^2+8\beta+1\right)y}{n^3\left(1-\beta\right)^7},\\
P_n^\beta\left(\left(\pi_1-\mathbf{1}x\right)^2\left(\pi_2-\mathbf{1}y\right)^2;x,y\right) =& \frac{\beta^4x^2y^2}{\left(1-\beta\right)^4}+\frac{\beta^2\left(x^2y+y^2x\right)}{n\left(1-\beta\right)^5}+\frac{xy}{n^2\left(1-\beta\right)^6}
\end{aligned}
$$

and

$$P_n^{\beta}\left(\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right]^2;x,y\right)=\frac{\beta^4\left(x^2+y^2\right)^2}{(1-\beta)^4}+\frac{2\beta^2\left[3\left(x^3+y^3\right)+x^2y+y^2x\right]}{n(1-\beta)^5}+\frac{\left[\left(8\beta^2+4\beta+3\right)\left(x^2+y^2\right)+2xy\right]}{n^2(1-\beta)^6}+\frac{\left(6\beta^2+8\beta+1\right)(x+y)}{n^3(1-\beta)^7}.$$

Proof. Proof can be obtained directly from Lemma 1.1. □

2. **Monotonicity of** $\left\{P_n^{\beta}(f)\right\}_{n\in\mathbb{N}}$**.** In [7], Cheney and Sharma proved that the Szász-Mirakjan operators $P_{n,1}^0(f)$ are non-increasing in n, when the attached function f is convex on $[0,\infty)$. In [6], the authors obtained the same result for a generalization of Jain operators in the sense of Schurer. In this section, we study the monotonicity of the sequence of bivariate Jain operators $P_n^{\beta}(f)$, $n\in\mathbb{N}$.

Below, we recall the notion of convexity of a real valued function f defined on the set $D\subset\mathbb{R}^2$.

Definition 2.1. A real valued continuous function on D is said to be convex in D *if*

$$f\left(\sum_{i=1}^{r}\alpha_i(x_i,y_i)\right)\leq\sum_{i=1}^{r}\alpha_i f(x_i,y_i)$$

for every $(x_1,y_1),\ldots,(x_r,y_r)\in D$ *and for every nonnegative numbers* $\alpha_1,\alpha_2,\ldots,\alpha_r$ *such that* $\alpha_1+\alpha_2+\cdots+\alpha_r=1$ (see, e.g., [5]).

We need the Abel type and Jensen's formulas, given by

$$(u+v+n\beta)^n=\sum_{i=0}^{n}\binom{n}{i}(u+i\beta)^i v\left[v+(n-i)\beta\right]^{n-i-1}\tag{6}$$

and

$$(u+v)(u+v+n\beta)^{n-1}=\sum_{i=0}^{n}\binom{n}{i}u(u+i\beta)^{i-1}v\left[v+(n-i)\beta\right]^{n-i-1},\tag{7}$$

respectively, where u, v and $\beta\in\mathbb{R}$, $n\in\mathbb{N}$ (see, e.g., [22]).

Theorem 2.2. *Let* $f\in C_B(D)$ *be convex. Then* $P_n^{\beta}(f)$, $0\leq\beta<1$, *is non-increasing in* n.

Proof. From the definition of the operators, it is enough to consider for x and y being positive. We can write

$$\begin{aligned}&P_n^{\beta}(f;x,y)-P_{n+1}^{\beta}(f;x,y)\\ =\;&e^xe^y\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{nx(nx+k\beta)^{k-1}}{k!}e^{-[(n+1)x+k\beta]}\\ &\times\frac{ny(ny+l\beta)^{l-1}}{l!}e^{-[(n+1)y+l\beta]}f\left(\frac{k}{n},\frac{l}{n}\right)\end{aligned}$$

$$-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1)x+k\beta]}$$
$$\times\frac{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{l!}e^{-[(n+1)y+l\beta]}f\left(\frac{k}{n+1},\frac{l}{n+1}\right).$$

From (2) and (3) we can write e^x and e^y as

$$e^x=\sum_{m=0}^{\infty}\frac{x\,(x+m\beta)^{m-1}}{m!}e^{-m\beta},\quad e^y=\sum_{p=0}^{\infty}\frac{y\,(y+p\beta)^{p-1}}{p!}e^{-p\beta}.$$

Making use of these expressions in the last formula, we obtain

$$\begin{aligned}&P_n^{\beta}(f;x,y)-P_{n+1}^{\beta}(f;x,y)\\ =\;&\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{x\,(x+m\beta)^{m-1}}{m!}e^{-m\beta}\frac{nx\,(nx+k\beta)^{k-1}}{k!}e^{-[(n+1)x+k\beta]}\\ &\times\frac{y\,(y+p\beta)^{p-1}}{p!}e^{-p\beta}\frac{ny\,(ny+l\beta)^{l-1}}{l!}e^{-[(n+1)y+l\beta]}f\left(\frac{k}{n},\frac{l}{n}\right)\\ &-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1)x+k\beta]}\\ &\times\frac{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{l!}e^{-[(n+1)y+l\beta]}f\left(\frac{k}{n+1},\frac{l}{n+1}\right).\end{aligned}$$

Replacing k with $k-m$ and l with $l-p$, then the above formula gives that

$$\begin{aligned}&P_n^{\beta}(f;x,y)-P_{n+1}^{\beta}(f;x,y)\\ =\;&\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=m}^{\infty}\sum_{l=p}^{\infty}\frac{x\,(x+m\beta)^{m-1}}{m!}e^{-m\beta}\\ &\frac{nx\,[nx+(k-m)\,\beta]^{k-m-1}}{(k-m)!}e^{-[(n+1)x+(k-m)\beta]}\\ &\times\frac{y\,(y+p\beta)^{p-1}}{p!}e^{-p\beta}\frac{ny\,[ny+(l-p)\,\beta]^{l-p-1}}{(l-p)!}e^{-[(n+1)y+(l-p)\beta]}f\left(\frac{k-m}{n},\frac{l-p}{n}\right)\\ &-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1)x+k\beta]}\\ &\times\frac{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{l!}e^{-[(n+1)y+l\beta]}f\left(\frac{k}{n+1},\frac{l}{n+1}\right).\end{aligned}$$

Interchanging the order of the summations, we get

$$\begin{aligned}&P_n^{\beta}(f;x,y)-P_{n+1}^{\beta}(f;x,y)\\ =\;&\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{l=0}^{\infty}\sum_{p=0}^{l}\frac{x\,(x+m\beta)^{m-1}}{m!}\frac{nx\,[nx+(k-m)\,\beta]^{k-m-1}}{(k-m)!}e^{-[(n+1)x+k\beta]}\\ &\times\frac{y\,(y+p\beta)^{p-1}}{p!}\frac{ny\,[ny+(l-p)\,\beta]^{l-p-1}}{(l-p)!}e^{-[(n+1)y+l\beta]}f\left(\frac{k-m}{n},\frac{l-p}{n}\right)\\ &-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1)x+k\beta]}\end{aligned}$$

$$\times \frac{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{l!}e^{-[(n+1)y+l\beta]}f\left(\frac{k}{n+1},\frac{l}{n+1}\right).$$

We can rewrite the last formula as

$$\begin{aligned}
&P_n^{\beta}(f;x,y)-P_{n+1}^{\beta}(f;x,y)\\
=&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}e^{-[(n+1)x+k\beta]}e^{-[(n+1)y+l\beta]}\\
&\times\left\{\sum_{m=0}^{k}\sum_{p=0}^{l}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{m!\,(k-m)!}\right.\\
&\times\frac{ny\,(ny+p\beta)^{p-1}\,y\,[y+(l-p)\,\beta]^{l-p-1}}{p!\,(l-p)!}f\left(\frac{m}{n},\frac{p}{n}\right)\\
&\left.-\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}{k!}\frac{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{l!}f\left(\frac{k}{n+1},\frac{l}{n+1}\right)\right\}\\
=&\;e^{-(n+1)(x+y)}\left[f(0,0)-f(0,0)\right]\\
&+e^{-(n+1)y}\sum_{k=1}^{\infty}e^{-[(n+1)x+k\beta]}\\
&\times\left\{\sum_{m=0}^{k}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{m!\,(k-m)!}f\left(\frac{m}{n},0\right)\right.\\
&\left.-\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}{k!}f\left(\frac{k}{n+1},0\right)\right\}\\
&+e^{-(n+1)x}\sum_{l=1}^{\infty}e^{-[(n+1)y+l\beta]}\\
&\times\left\{\sum_{p=0}^{l}\frac{ny\,(ny+p\beta)^{p-1}}{p!}\frac{y\,[y+(l-p)\,\beta]^{l-p-1}}{(l-p)!}f\left(0,\frac{p}{n}\right)\right.\\
&\left.-\frac{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{l!}f\left(0,\frac{l}{n+1}\right)\right\}\\
&+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}e^{-[(n+1)x+k\beta]}e^{-[(n+1)y+l\beta]}\\
&\times\left\{\sum_{m=0}^{k}\sum_{p=0}^{l}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{m!\,(k-m)!}\right.\\
&\times\frac{ny\,(ny+p\beta)^{p-1}\,y\,[y+(l-p)\,\beta]^{l-p-1}}{p!\,(l-p)!}f\left(\frac{m}{n},\frac{p}{n}\right)\\
&\left.-\frac{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}\,(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}{k!l!}f\left(\frac{k}{n+1},\frac{l}{n+1}\right)\right\}\\
=&\;I_1+I_2+I_3,
\end{aligned}$$

where

$$
\begin{aligned}
I_1 \quad &: \quad = e^{-(n+1)y} \sum_{k=1}^{\infty} \omega_\beta \left(k, (n+1)\, x\right) \\
&\times \left\{ \sum_{m=0}^{k} \binom{k}{m} \frac{nx \left(nx + m\beta\right)^{m-1} x \left[x + (k-m)\, \beta\right]^{k-m-1}}{(n+1)\, x \left[(n+1)\, x + k\beta\right]^{k-1}} f\left(\frac{m}{n}, 0\right) \right. \\
&\quad \left. - f\left(\frac{k}{n+1}, 0\right) \right\}, \\
I_2 \quad &: \quad = e^{-(n+1)x} \sum_{l=1}^{\infty} \omega_\beta \left(l, (n+1)\, y\right) \\
&\times \left\{ \sum_{p=0}^{l} \binom{l}{p} \frac{ny \left(ny + p\beta\right)^{p-1} y \left[y + (l-p)\, \beta\right]^{l-p-1}}{(n+1)\, y \left[(n+1)\, y + l\beta\right]^{l-1}} f\left(0, \frac{p}{n}\right) - f\left(0, \frac{l}{n+1}\right) \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 \quad &: \quad = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \omega_\beta \left(k, (n+1)\, x\right) \omega_\beta \left(l, (n+1)\, y\right) \\
&\times \left\{ \sum_{m=0}^{k} \sum_{p=0}^{l} \binom{k}{m} \binom{l}{p} \frac{nx \left(nx + m\beta\right)^{m-1} x \left[x + (k-m)\, \beta\right]^{k-m-1}}{(n+1)\, x \left[(n+1)\, x + k\beta\right]^{k-1}} \right. \\
&\quad \left. \times \frac{ny \left(ny + p\beta\right)^{p-1} y \left[y + (l-p)\, \beta\right]^{l-p-1}}{(n+1)\, y \left[(n+1)\, y + l\beta\right]^{l-1}} f\left(\frac{m}{n}, \frac{p}{n}\right) - f\left(\frac{k}{n+1}, \frac{l}{n+1}\right) \right\},
\end{aligned}
$$

with ω_β is given by (2). Now, it only remains to show that I_1, I_2 and I_3 are non-negative.

For I_1, we take

$$
\alpha_m := \binom{k}{m} \frac{nx \left(nx + m\beta\right)^{m-1} x \left[x + (k-m)\, \beta\right]^{k-m-1}}{(n+1)\, x \left[(n+1)\, x + k\beta\right]^{k-1}} \geq 0
$$

and

$$
\left(\frac{m}{n}, 0\right) \in D,
$$

where $m = 0, 1, \ldots, k$. Therefore, taking $u = nx,\ v = x,\ n = k$ and $i = m$ in (7), we obtain

$$
\begin{aligned}
&(n+1)\, x \left[(n+1)\, x + k\beta\right]^{k-1} \\
&= \sum_{m=0}^{k} \binom{k}{m} nx \left(nx + m\beta\right)^{m-1} x \left[x + (k-m)\, \beta\right]^{k-m-1},
\end{aligned}
$$

which gives that

$$
\sum_{m=0}^{k} \alpha_m = 1.
$$

On the other hand, by using the well known properties of multiplication by scalar and addition of vectors, we deduce

$$
\sum_{m=0}^{k} \alpha_m \left(\frac{m}{n}, 0\right)
$$

$$
\begin{aligned}
&= \left(\sum_{m=1}^{k}\binom{k}{m}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}\frac{m}{n},0\right)\\
&= \left(\sum_{m=0}^{k-1}\binom{k-1}{m}\frac{knx\,(nx+\beta+m\beta)^{m}\,x\,[x+(k-m-1)\,\beta]^{k-m-2}}{n\,(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}},0\right)\\
&= \left(\frac{k}{n+1}\sum_{m=0}^{k-1}\binom{k-1}{m}\frac{(nx+\beta+m\beta)^{m}\,x\,[x+(k-m-1)\,\beta]^{k-m-2}}{[(n+1)\,x+k\beta]^{k-1}},0\right)\\
&= \left(\frac{k}{n+1},0\right),
\end{aligned}
$$

where we have used the formula (6), by taking $u = nx+\beta$, $v = x$, $i = m$ and $n = k-1$, which gives that

$$
[(n+1)\,x+k\beta]^{k-1} = \sum_{m=0}^{k-1}\binom{k-1}{m}(nx+\beta+m\beta)^{m}\,x\,[x+(k-m-1)\,\beta]^{k-m-2}.
$$

Therefore, we obtain

$$
\sum_{m=0}^{k}\binom{k}{m}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}f\left(\frac{m}{n},0\right) \geq f\left(\frac{k}{n+1},0\right),
$$

which is a direct consequence of the convexity of f. That is, we get $I_1 \geq 0$. The case $I_2 \geq 0$ is obtained in a similar way by taking

$$
\gamma_p = \binom{l}{p}\frac{ny\,(ny+p\beta)^{p-1}\,y\,[y+(l-p)\,\beta]^{l-p-1}}{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}} \geq 0
$$

and

$$
\left(0,\frac{p}{n}\right) \in D,
$$

where $p = 0,1,\ldots,l$. Finally, for I_3, taking

$$
\alpha_m\sum_{p=0}^{l}\gamma_p \geq 0,\ \ m = 0,1,\ldots,k,
$$

as the non-negative constants such that $\sum_{m=0}^{k}\left(\alpha_m\sum_{p=0}^{l}\gamma_p\right) = 1$, where α_m and γ_p are the same as the given above, and $\left(\frac{m}{n},\frac{p}{n}\right) \in D$. From the convexity of f (see Definition 2.1) and using the same calculations as above for each component, we arrive at

$$
\begin{aligned}
&\sum_{m=0}^{k}\binom{k}{m}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}\\
&\times\sum_{p=0}^{l}\binom{l}{m}\frac{ny\,(ny+p\beta)^{p-1}\,y\,[y+(l-p)\,\beta]^{l-p-1}}{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}f\left(\frac{m}{n},\frac{p}{n}\right)\\
\geq\ & f\left(\sum_{m=0}^{k}\sum_{p=0}^{l}\binom{k}{m}\binom{l}{m}\frac{nx\,(nx+m\beta)^{m-1}\,x\,[x+(k-m)\,\beta]^{k-m-1}}{(n+1)\,x\,[(n+1)\,x+k\beta]^{k-1}}\right.\\
&\left.\times\frac{ny\,(ny+p\beta)^{p-1}\,y\,[y+(l-p)\,\beta]^{l-p-1}}{(n+1)\,y\,[(n+1)\,y+l\beta]^{l-1}}\left(\frac{m}{n},\frac{p}{n}\right)\right)
\end{aligned}
$$

$$= f\left(\frac{k}{n+1}, \frac{l}{n+1}\right),$$

which shows that $I_3 \geq 0$. This completes the proof. □

3. **A retaining property.** For the well known Bernstein polynomials, B_n, $n \in \mathbb{N}$, Li [16] proved that *if ω is a modulus of continuity, then for each n, $B_n(\omega)$ is also a modulus of continuity.* The same result proved for multivariate non tensor product Baskakov operators by Cao et al. [5]. Now, we recall the definition of function of modulus of continuity in two variables.

Definition 3.1. A continuous and non-negative function ω defined on D is called a modulus of continuity, if it satisfies the following conditions for any (x, y), $(t, s) \in D$:

1. $\omega(0, 0) = 0$,
2. $\omega(x, y)$ *is non-decreasing (in each coordinate), namely,* $\omega(x, y) \geq \omega(t, s)$ for $(x, y) \geq (t, s)$,
3. $\omega(x, y)$ *is sub-additive, i.e.;* $\omega(x + t, y + s) \leq \omega(x, y) + \omega(t, s)$,

(see [5]).

Here, we present the same result for bivariate Jain operators.

Theorem 3.2. *Let ω be a modulus of continuity and $0 \leq \beta < 1$. Then $P_n^{\beta}(\omega)$ is also a modulus of continuity for each $n \in \mathbb{N}$.*

Proof. Let (x, y), $(t, s) \in D$ and $(x, y) \leq (t, s)$. In this case, by (4) we have

$$P_n^{\beta}(\omega; t, s) \tag{8}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{nt(nt+i\beta)^{i-1}}{i!} e^{-(nt+i\beta)} \frac{ns(ns+j\beta)^{j-1}}{j!} e^{-(ns+j\beta)} \omega\left(\frac{i}{n}, \frac{j}{n}\right).$$

Taking $u = nx$, $v = nt - nx$ and $n = i$ and $i = k$ in (7), we get

$$nt(nt+i\beta)^{i-1} = \sum_{k=0}^{i} \binom{i}{k} nx(nx+k\beta)^{k-1} n(t-x)[n(t-x)+(i-k)\beta]^{i-k-1}$$

and similarly, taking $u = ny$, $v = ns - ny$ and $n = j$ and $i = l$, we get

$$ns(ns+j\beta)^{j-1} = \sum_{l=0}^{j} \binom{j}{l} ny(ny+l\beta)^{l-1} n(s-y)[n(s-y)+(j-l)\beta]^{j-l-1}.$$

Making use of these expressions in (8) we obtain

$$P_n^{\beta}(\omega; t, s)$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j} \binom{i}{k} \frac{1}{i!} nx(nx+k\beta)^{k-1} n(t-x)[n(t-x)+(i-k)\beta]^{i-k-1} e^{-(nt+i\beta)}$$
$$\times \binom{j}{l} \frac{1}{j!} ny(ny+l\beta)^{l-1} n(s-y)[n(s-y)+(j-l)\beta]^{j-l-1} e^{-(ns+j\beta)} \omega\left(\frac{i}{n}, \frac{j}{n}\right).$$

Interchanging the order of the above summations and then taking $i-k = p$, $j-l = m$ in the result, we obtain

$$P_n^{\beta}(\omega; t, s) \tag{9}$$
$$= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\sum_{l=0}^{\infty}\sum_{j=l}^{\infty} \frac{nx(nx+k\beta)^{k-1}}{k!} \frac{n(t-x)[n(t-x)+(i-k)\beta]^{i-k-1}}{(i-k)!} e^{-(nt+i\beta)}$$

$$\times\frac{ny(ny+l\beta)^{l-1}}{l!}\frac{n(s-y)[n(s-y)+(j-l)\beta]^{j-l-1}}{(j-l)!}e^{-(ns+j\beta)}\omega\left(\frac{i}{n},\frac{j}{n}\right)$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{nx(nx+k\beta)^{k-1}}{k!}\frac{n(t-x)[n(t-x)+p\beta]^{p-1}}{p!}e^{-[nt+(k+p)\beta)]}$$
$$\times\frac{ny(ny+l\beta)^{l-1}}{l!}\frac{n(s-y)[n(s-y)+m\beta]^{m-1}}{m!}e^{-[ns+(l+m)\beta)]}\omega\left(\frac{k+p}{n},\frac{l+m}{n}\right).$$

On the other hand, $P_n^{\beta}(\omega;x,y)$ can be written as

$$P_n^{\beta}(\omega;x,y)$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{nx(nx+k\beta)}{k!}^{k-1}e^{-[n(t+x-t)+k\beta]}$$
$$\times\frac{ny(ny+l\beta)}{l!}^{l-1}e^{-[n(s+y-s)+l\beta]}\omega\left(\frac{k}{n},\frac{l}{n}\right).$$

Here, from (2) and (3) we can write

$$e^{n(t-x)}=\sum_{p=0}^{\infty}\frac{n(t-x)\left[n(t-x)+p\beta\right]^{p-1}}{p!}e^{-p\beta}$$

and

$$e^{n(s-y)}=\sum_{m=0}^{\infty}\frac{n(s-y)\left[n(s-y)+m\beta\right]^{m-1}}{m!}e^{-m\beta}.$$

Substituting these expressions in the last formula, we obtain

$$P_n^{\beta}(\omega;x,y) \tag{10}$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{nx(nx+k\beta)^{k-1}}{k!}\frac{n(t-x)[n(t-x)+p\beta]^{p-1}}{p!}e^{-[nt+(k+p)\beta]}$$
$$\times\frac{ny(ny+l\beta)^{l-1}}{l!}\frac{n(s-y)[n(s-y)+m\beta]^{m-1}}{m!}e^{-[ns+(l+m)\beta]}\omega\left(\frac{k}{n},\frac{l}{n}\right).$$

Subtracting (10) from (9), we deduce

$$P_n^{\beta}(\omega;t,s)-P_n^{\beta}(\omega;x,y) \tag{11}$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{nx(nx+k\beta)^{k-1}}{k!}\frac{n(t-x)[n(t-x)+p\beta]^{p-1}}{p!}e^{-[nt+(k+p)\beta]}$$
$$\times\frac{ny(ny+l\beta)^{l-1}}{l!}\frac{n(s-y)[n(s-y)+m\beta]^{m-1}}{m!}e^{-[ns+(l+m)\beta]}$$
$$\times\left\{\omega\left(\frac{k+p}{n},\frac{l+m}{n}\right)-\omega\left(\frac{k}{n},\frac{l}{n}\right)\right\}.$$

Using the sub-additivity of ω, we arrive at

$$P_n^{\beta}(\omega;t,s)-P_n^{\beta}(\omega;x,y)$$
$$\leq e^{nx}\sum_{p=0}^{\infty}\frac{n(t-x)[n(t-x)+p\beta]^{p-1}}{p!}e^{-(nt+p\beta)}$$
$$\times e^{ny}\sum_{m=0}^{\infty}\frac{n(s-y)[n(s-y)+m\beta]^{m-1}}{m!}e^{-(ns+m\beta)}\omega\left(\frac{p}{n},\frac{m}{n}\right)$$

$$
\begin{aligned}
&= \sum_{p=0}^{\infty}\sum_{m=0}^{\infty} \omega_\beta(p, n(t-x))\omega_\beta(m, n(s-y))\,\omega\left(\frac{p}{n}, \frac{m}{n}\right)\\
&= P_n^\beta(\omega; t-x, s-y),
\end{aligned}
$$

where ω_β is given by (2). This shows that $P_n^\beta(\omega)$ is sub-additive. Since the modulus of continuity ω is non-decreasing, (11) provides that $P_n^\beta(\omega; t, s) \geq P_n^\beta(\omega; x, y)$ when $(t, s) \geq (x, y)$. Hence, $P_n^\beta(\omega)$ is non-decreasing. From the definition of the operators $P_n^\beta(\omega)$, it is obvious that

$$\lim_{(x,y)\to(0,0)} P_n^\beta(\omega; x, y) = P_n^\beta(\omega; 0, 0) = \omega(0, 0) = 0.$$

Thereby, $P_n^\beta(\omega)$ itself is a function of modulus of continuity for each $n \in \mathbb{N}$. □

In this theorem, the case $\beta = 0$ gives the corresponding result for the bivariate Szász–Mirakjan operators in [18].

4. **A Voronovskaja type result.** In [11], Farcaş, established the following Voronovskaja type result for univariate Jain operators

$$\lim_{n\to\infty} n\left(P_{n,1}^{\beta_n}(f; x) - f(x)\right) = \frac{x}{2}f''(x),\ x > 0,$$

for $f \in C^2[0, \infty)$, the space of all continuous functions having continuous second order derivative, where $\beta = \beta_n,\ 0 \leq \beta_n < 1$, is a sequence such that $\lim_{n\to\infty} n\beta_n = 0$. Later, under the condition that $\lim_{n\to\infty} n\beta_n = \gamma$ is finite, Abel and Agratini [1, Corollary 1] gave the following Voronovskaja type result for the Jain operators

$$\lim_{n\to\infty} n\left(P_{n,1}^{\beta_n}(f; x) - f(x)\right) = \gamma f'(x) + \frac{x}{2}f''(x)$$

for locally bounded functions on $(0, \infty)$ satisfying the growth condition $f(x) = O(x^2)$ as $x \to \infty$. Olgun et al. [19] obtained a Voronovskaja type result for a generalization of the Jain operators. In this part, we present a Voronovskaja type theorem for the sequence of bivariate Jain operators $P_n^{\beta_n},\ n \in \mathbb{N}$.

Theorem 4.1. *Let $f \in C_B(D)$ admit its partial derivatives up to the second order at $(x, y) \in K$, where K is a compact subset of D, and assume that $\lim_{n\to\infty} n\beta_n = 0,\ 0 \leq \beta_n < 1$. Then we have*

$$\lim_{n\to\infty} n\left\{P_n^{\beta_n}(f; x, y) - f(x, y)\right\} = \frac{x}{2}f_{xx}(x, y) + \frac{y}{2}f_{yy}(x, y).$$

Proof. By Taylor's formula of f at the fixed point $(x, y) \in K$, we have

$$
\begin{aligned}
& f(t, s)\\
= \ & f(x, y) + f_x(x, y)(t-x) + f_y(x, y)(s-y)\\
& + \frac{1}{2}\left\{f_{xx}(x, y)(t-x)^2 + 2f_{xy}(x, y)(t-x)(s-y) + f_{yy}(x, y)(s-y)^2\right\}\\
& + \lambda(t-x, s-y)\left[(t-x)^2 + (s-y)^2\right],
\end{aligned}
$$

for all $(t, s) \in D$, where λ is bounded on D and $\lim_{(u,v)\to(0,0)} \lambda(u, v) = 0$ (see, e.g., [4]). λ can be made continuous at $(u, v) = (0, 0)$ by defining $\lambda(0, 0) = 0$. Thus, for any given $\varepsilon > 0$, there is a $\delta > 0$ such that $|\lambda(t-x, s-y)| < \varepsilon$ whenever $\sqrt{(t-x)^2 + (s-y)^2} < \delta$ and $(t, s) \in D$. On the other hand, since λ is bounded, there is an $M > 0$ such that $|\lambda(u, v)| \leq M$ for all $(u, v) \in D$. Therefore, we

can write that $|\lambda(t-x,s-y)| \leq \frac{M}{\delta^2}\left[(t-x)^2+(s-y)^2\right]$ when $\sqrt{(t-x)^2+(s-y)^2} \geq \delta$. Taking into account of these facts, we can write

$$\begin{aligned} & |\lambda(t-x,s-y)|\left[(t-x)^2+(s-y)^2\right] \\ \leq \quad & \varepsilon\left[(t-x)^2+(s-y)^2\right]+\frac{M}{\delta^2}\left[(t-x)^2+(s-y)^2\right]^2 \end{aligned} \tag{12}$$

for the given (x,y), and all $(t,s) \in D$. Now, Applying the operators $P_n^{\beta_n}$ to the both sides of the Taylor formula given above, from Lemma 1.2, we have

$$\begin{aligned} & P_n^{\beta_n}(f;x,y)-f(x,y) \\ = \quad & f_x(x,y)\frac{\beta_n x}{1-\beta_n}+f_y(x,y)\frac{\beta_n y}{1-\beta_n} \\ & +\frac{f_{xx}(x,y)}{2}\left[\frac{\beta_n^2 x^2}{(1-\beta_n)^2}+\frac{x}{n(1-\beta_n)^3}\right]+f_{xy}(x,y)\frac{\beta_n^2 xy}{(1-\beta_n)^2} \\ & +\frac{f_{yy}(x,y)}{2}\left[\frac{\beta_n^2 y^2}{(1-\beta_n)^2}+\frac{y}{n(1-\beta_n)^3}\right] \\ & +P_n^{\beta_n}\left(\lambda(\pi_1-\mathbf{1}x,\pi_2-\mathbf{1}y)\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right];x,y\right). \end{aligned}$$

Using the hypothesis that $\lim_{n\to\infty} n\beta_n = 0$, we obtain

$$\begin{aligned} & \lim_{n\to\infty} n\left[P_n^{\beta_n}(f;x,y)-f(x,y)\right] \\ = \quad & \frac{x}{2}f_{xx}(x,y)+\frac{y}{2}f_{yy}(x,y) \\ & +\lim_{n\to\infty} nP_n^{\beta_n}\left(\lambda(\pi_1-\mathbf{1}x,\pi_2-\mathbf{1}y)\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right];x,y\right). \end{aligned}$$

Now, taking into account of (12) and monotonicity of the operator $P_n^{\beta_n}$, we have

$$\begin{aligned} 0 \quad \leq \quad & \left|P_n^{\beta_n}\left(\lambda(\pi_1-\mathbf{1}x,\pi_2-\mathbf{1}y)\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right];x,y\right)\right| \\ \leq \quad & \varepsilon P_n^{\beta_n}\left(\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right];x,y\right) \\ & +\frac{M}{\delta^2}P_n^{\beta_n}\left(\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right]^2;x,y\right). \end{aligned}$$

Thus, from Lemma 1.2 and the hypothesis on β_n, we reach to

$$\lim_{n\to\infty} nP_n^{\beta_n}\left(\lambda(\pi_1-\mathbf{1}x,\pi_2-\mathbf{1}y)\left[(\pi_1-\mathbf{1}x)^2+(\pi_2-\mathbf{1}y)^2\right];x,y\right)=0$$

for $(x,y) \in K$, which completes the proof. □

In this theorem, the case $\beta = 0$ gives the corresponding result for the bivariate Szász–Mirakjan operators (see, Theorem 1 in [21], with $p_1 = p_2 = 0$).

Acknowledgments. The authors are grateful to the anonymous reviewers for valuable suggestions which greatly improved this paper.

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Received June 2021; revised September 2021; early access October 2021.

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