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Research Article

d-Orthogonality of q-Sheffer Polynomials

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ABSTRACT

In this paper, we find necessary and sufficient conditions for q-Sheffer polynomials to be d-orthogonal. Moreover, an example is presented and with the help of this example, we rediscover some known d-orthogonal polynomial sets.

Keywords: $d-$Orthogonal polynomials, q-Sheffer polynomials, generating function, orthogonal polynomials
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1 Introduction

In the past two decades, there has been a growing interest on the notion of d-orthogonal polynomials which are subclass of multiple orthogonal polynomials of type II [1–4]. d-orthogonal polynomials are natural extensions of orthogonal polynomials. So far, many new d-orthogonal polynomials have been presented as solutions of characterization problems by various authors [5–22, 24–26, 29–31, 36–38]. In this study, we investigate the d-orthogonality of q-Sheffer polynomials. First of all, let us give some basic and brief informations about d-orthogonal polynomials.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and $\mathcal{P}'$ be its algebraic dual. The effect of any linear functional $u \in \mathcal{P}'$ to the polynomial $f \in \mathcal{P}$ is denoted by $\langle u, f \rangle$. A polynomial sequence $\{P_n\}_{n\geq 0} \in \mathcal{P}$ is called a polynomial set if $\deg P_n = n$ for all integer n.

Definition 1. *The polynomial set $\{P_n\}_{n\geq 0}$ is said to be d-orthogonal polynomial set with respect to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, ..., u_{d-1})$ if it holds*

$$\begin{cases} \langle u_k, P_m P_n \rangle = 0, & m > nd + k,\ n \geq 0, \\ \langle u_k, P_n P_{nd+k} \rangle \neq 0, & n \geq 0, \end{cases} \tag{1}$$

for $d \in \mathbb{Z}^+$ and $k \in \{0, 1, ..., d-1\}$ [32, 35].

The above orthogonality conditions (1) are equivalent to the fact that the polynomial set $\{P_n\}_{n\geq 0}$ satisfies the following recurrence relation in the monic form

$$P_{n+1}(x) = (x - \delta_{n+1})P_n(x) + \sum_{k=1}^{d} \lambda_k^n P_{n-k}(x), \quad n \geq 0, \tag{2}$$

with $\lambda_d^n \neq 0$, $n \geq d$ and $P_{-n}(x) = 0$, $n \geq 1$ [32]. Note that when $d = 1$, the d-orthogonality conditions (1) and the $(d+1)$-order recurrence relations (2) reduce to the celebrated orthogonality conditions and three-term recurrence relations.

Next, let us give some definitions and formulas about q-calculus [28]. The q-analogue of any complex number n is given by

$$[n]_q = \frac{1-q^n}{1-q}, \quad q \in \mathbb{C}\setminus\{0,1\}.$$

q-shifted factorial, q-factorial and q-binomial coefficients are defined by the following equalities for $0 < |q| < 1$

$$\begin{aligned} (a;q)_n &= \prod_{i=0}^{n-1}(1-aq^i), \\ n!_q &= [n]_q\,[n-1]_q \ldots [1]_q, \quad 0!_q = 1, \\ \left[\begin{array}{c} n \\ k \end{array}\right]_q &= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{n!_q}{k!_q(n-k)!_q}. \end{aligned}$$

Also, the q-analogue of exponential function is represented by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!_q} = \frac{1}{((1-q)x;q)_\infty} \tag{3}$$

where $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$ and $0 < |q| < 1$. q-derivative of a given function f is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

and the product rule

$$D_q(gf)(x) = g(qx)D_q\left[f(x)\right] + D_q\left[g(x)\right] f(x). \tag{4}$$

Now, we introduce the form of Eulerian generating function and q-composition of two functions [23, 27]. The set of the Eulerian generating function $\mathcal{E}_q(n)$ is in the following form

$$a_n\frac{t^n}{n!_q} + a_{n+1}\frac{t^{n+1}}{(n+1)!_q} + a_{n+2}\frac{t^{n+2}}{(n+2)!_q} + \ldots, \ (a_n = 1),$$

where $n \in \mathbb{N}_0$. If $f \in \mathcal{E}_q(1)$, then the $k-$th symbolic power $f^{[k]}$ of f is inductively given by

$$D_q f^{[k]}(t) = [k]_q\, f^{[k-1]}(t) D_q\left[f(t)\right]. \tag{5}$$

Here $k \geq 1$, $f^{[0]}(t) = 1$ and $f^{[1]}(t) = f(t)$. The q-composition of g with f is defined by

$$g\,[f] = \sum_{n=0}^{\infty} g_n \frac{f^{[n]}(t)}{n!_q}, \tag{6}$$

where $g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!_q} \in \mathcal{E}_q(0)$. Moreover, we obtain $g\,[t] = g(t)$ since $t^{[n]} = t^n$.

The q-Sheffer polynomial sequence can be defined with the help of the following generating function of the form

$$ge_q\,[xf] = \sum_{n=0}^{\infty} s_n^{(q)}(x) \frac{t^n}{n!_q}, \tag{7}$$

where $g \in \mathcal{E}_q(0)$ and $f \in \mathcal{E}_q(1)$ [23]. It is worthy to note that whenever $q \to 1^-$, we face with the well-known Sheffer polynomial sequence. In this paper, first, we investigate the d-orthogonality of the q-Sheffer polynomial sequence given by (7). We obtain necessary and sufficient conditions for q-Sheffer polynomial sequence to be d-orthogonal. Moreover, an example is given as the solution of this characterization problem.

2 d-Orthogonality Condition for q-Sheffer Polynomials

In this section, we will prove our main result. We will obtain necessary and sufficient conditions for the d-orthogonality of q-Sheffer polynomials generated by (7). Note that, taking the q-derivative of both sides of the equality (6) in view of (5), it is easy to see that the q-derivative of q-composition of two functions will be [27]

$$D_q g\,[f] = (D_q g)\,[f]\,(D_q f). \tag{8}$$

The equality (8) can be seen as the q-analogue of the chain rule which will be used in the proof of next theorem.

Theorem 1. *The polynomial set* $\left\{s_n^{(q)}\right\}_{n\geq 0}$ *generated by* (7) *is d-orthogonal if and only if* $g \in \mathcal{E}_q(0)$ *and* $f \in \mathcal{E}_q(1)$ *satisfy the following conditions*

$$\frac{D_q g(t)}{g(t)} = -\frac{\sum_{k=0}^{d} \beta_k t^k}{1 + \sum_{k=1}^{d+1} \alpha_k t^k}, \quad D_q f(t) = \frac{g(t)}{\left[1 + \sum_{k=1}^{d+1} \alpha_k t^k\right] g(qt)} \tag{9}$$

and also $\beta_d + [n-d]_q\, \alpha_{d+1} \neq 0$.

Proof. By using (4), taking q-derivative of both sides of (7) with respect to t leads us to

$$g(qt) D_q\left(e_q\,[xf]\right) + \left(D_q g(t)\right) e_q\,[xf] = \sum_{n=0}^{\infty} s_{n+1}^{(q)}(x) \frac{t^n}{n!_q}$$

where we use the fact $D_q t^n = [n]_q\, t^{n-1}$ on the right-hand side of the above equality. If we take the chain

rule given with (8) into account, we obtain

$$g(qt)x\left(D_q f(t)\right)e_q\left[xf\right]+\left(D_q g(t)\right)e_q\left[xf\right]=\sum_{n=0}^{\infty}s_{n+1}^{(q)}(x)\frac{t^n}{n!_q}.$$

After some manipulations, we reach

$$\sum_{n=0}^{\infty}s_{n+1}^{(q)}(x)\frac{t^n}{n!_q}=\left[\frac{g(qt)(D_q f(t))}{g(t)}x+\frac{(D_q g(t))}{g(t)}\right]g(t)e_q\left[xf\right]. \tag{10}$$

If we combine the conditions given by (9) with (10), we get

$$\sum_{n=0}^{\infty}s_{n+1}^{(q)}(x)\frac{t^n}{n!_q}=\left[\frac{x}{1+\sum\limits_{k=1}^{d+1}\alpha_k t^k}-\frac{\sum\limits_{k=0}^{d}\beta_k t^k}{1+\sum\limits_{k=1}^{d+1}\alpha_k t^k}\right]\sum_{n=0}^{\infty}s_n^{(q)}(x)\frac{t^n}{n!_q}$$

and so

$$\left[1+\sum_{k=1}^{d+1}\alpha_k t^k\right]\sum_{n=0}^{\infty}s_{n+1}^{(q)}(x)\frac{t^n}{n!_q}=\left[x-\sum_{k=0}^{d}\beta_k t^k\right]\sum_{n=0}^{\infty}s_n^{(q)}(x)\frac{t^n}{n!_q}.$$

From the last equality, after some calculations and shifting indices, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}s_{n+1}^{(q)}(x)\frac{t^n}{n!_q}+\sum_{n=0}^{\infty}\left[\sum_{k=1}^{d+1}\alpha_k k!_q\begin{bmatrix}n\\k\end{bmatrix}_q s_{n+1-k}^{(q)}(x)\right]\frac{t^n}{n!_q}\\ =\;&\sum_{n=0}^{\infty}x s_n^{(q)}(x)\frac{t^n}{n!_q}-\sum_{n=0}^{\infty}\left[\sum_{k=0}^{d}\beta_k k!_q\begin{bmatrix}n\\k\end{bmatrix}_q s_{n-k}^{(q)}(x)\right]\frac{t^n}{n!_q}.\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!_q}$, we conclude that the following recurrence relation again by shifting indices

$$s_{n+1}^{(q)}(x)=\left[x-(\beta_0+\alpha_1\left[n\right]_q)\right]s_n^{(q)}(x)-\sum_{k=1}^{d}\left[\beta_k\begin{bmatrix}n\\k\end{bmatrix}_q+\alpha_{k+1}\left[k+1\right]_q\begin{bmatrix}n\\k+1\end{bmatrix}_q\right]k!_q s_{n-k}^{(q)}(x).$$

The above recurrence relation satisfied by q-Sheffer polynomial set $\left\{s_n^{(q)}\right\}_{n\geq 0}$ is in the form of (2). Thus, this means that q-Sheffer polynomial set $\left\{s_n^{(q)}\right\}_{n\geq 0}$ generated by (7) is d-orthogonal with the following regularity conditions

$$\beta_d+\left[n-d\right]_q\alpha_{d+1}\neq 0.$$

Conversely, if the q-Sheffer polynomial set $\left\{s_n^{(q)}\right\}_{n\geq 0}$ is d-orthogonal, it is easy to obtain the conditions given with (9) by above calculations. □

Remark 1. *For $d=1$, we meet the orthogonality of q-Sheffer polynomials characterised by Cheon and Jung [23]. In [12], Cheikh and Gam treated the limiting case $q\to 1^-$ for general positive integer d and*

obtained d-orthogonality of Sheffer polynomials. Also, by taking $d = 1$ and $q \to 1^-$, Meixner and Sheffer found all of the form of the orthogonal Sheffer polynomials in [33] and [34].

Example. Let us assume that $\alpha_k - (q-1)\beta_{k-1} = 0$ for $1 \leq k \leq d+1$ and $\beta_d \neq 0$. In this case, by using the q-derivative of $g(t)$ and the conditions (9), it becomes

$$\frac{g(qt)}{g(t)} = \frac{1}{1 + \sum_{k=1}^{d+1} \alpha_k t^k} \tag{11}$$

and thus

$$D_q f(t) = 1.$$

So, we obtain $f(t) = t$. From (11), it can be written as

$$g(t) = (1-(1-q)x_0 t)(1-(1-q)x_1 t)...(1-(1-q)x_d t)g(qt)$$

where $x_0, x_1, ..., x_d \in \mathbb{C}/\{0\}$. In view of (3), iterating the last equality leads us to

$$g(t) = \frac{1}{e_q(x_0 t)...e_q(x_d t)}.$$

This means that we recover the generating form of q-Appell polynomial sets, which are at the same time d-orthogonal, obtained by Zaghouani [38].

Moreover, if $q \to 1^-$, the above assumptions turn out to be

$$\alpha_k = 0 \text{ and } \beta_d \neq 0$$

where $1 \leq k \leq d+1$. In this case, q-derivative operator becomes the derivative operator and so we derive the following conditions from (9)

$$\frac{g'(t)}{g(t)} = -\sum_{k=0}^{d} \beta_k t^k \text{ and } f'(t) = 1.$$

Integrating both sides gives

$$g(t) = \exp(H_{d+1}(t)) \text{ and } f(t) = t.$$

Here, $H_{d+1}(t)$ is a polynomial of degree $d+1$ such that the leading coefficient is $\frac{-\beta_d}{d+1}$. Indeed, we meet the characterization of d-orthogonality of Appell polynomials which was founded by Douak in [24].

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The author declare that no generative Artificial Intelligence tools were used in the preparation of this manuscript.

References

[1] Aptekarev, A. I. (1998). Multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 99:423–447.

[2] Aptekarev, A. I., Branquinho, A., and Van Assche, W. (2003). Multiple orthogonal polynomials for classical weights. *Transactions of the American Mathematical Society*, 355:3887–3914.

[3] Aptekarev, A. I., Marcellán, F., and Rocha, I. A. (1997). Semi classical multiple orthogonal polynomials and the properties of Jacobi-Bessel polynomials. *Journal of Approximation Theory*, 90:117–146.

[4] Arvesú, J., Coussement, J., and Van Assche, W. (2003). Some discrete multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 153:19–45.

[5] Ben Cheikh, Y. and Ben Romdhane, N. (2011). On d-symmetric classical d-orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 236:85–93.

[6] Ben Cheikh, Y. and Ben Romdhane, N. (2014). d-symmetric d-orthogonal polynomials of Brenke type. *Journal of Mathematical Analysis and Applications*, 416:735–747.

[7] Ben Cheikh, Y. and Douak, K. (2000). On the classical d-orthogonal polynomials defined by certain generating functions i. *Bulletin of the Belgian Mathematical Society*, 7:107–124.

[8] Ben Cheikh, Y. and Douak, K. (2001). On the classical d-orthogonal polynomials defined by certain generating functions ii. *Bulletin of the Belgian Mathematical Society*, 8:591–605.

[9] Ben Cheikh, Y. and Gaied, M. (2007). Dunkl-Appell d-orthogonal polynomials. *Integral Transforms and Special Functions*, 18:581–597.

[10] Ben Cheikh, Y. and Gaied, M. (2010). A Dunkl-classical d-symmetric d-orthogonal polynomial set. *Rendiconti di Matematica e delle sue Applicazioni*, 30:195–219.

[11] Ben Cheikh, Y. and Gaied, M. (2011). A characterization of Dunkl-classical d-symmetric d-orthogonal polynomials and its applications. *Journal of Computational and Applied Mathematics*, 236:49–64.

[12] Ben Cheikh, Y. and Gam, I. (2016). On some operators varying the dimensional parameters of d-orthogonality. *Integral Transforms and Special Functions*, 27:731–746.

[13] Ben Cheikh, Y. and Gam, I. (2019). L-classical d-orthogonal polynomial sets of Sheffer type. *Filomat*, 33:881–895.

[14] Ben Cheikh, Y., Lamiri, I., and Ouni, A. (2009). On Askey-scheme and d-orthogonality. i. a characterization theorem. *Journal of Computational and Applied Mathematics*, 233:621–629.

[15] Ben Cheikh, Y. and Ouni, A. (2008). Some generalized hypergeometric d-orthogonal polynomial sets. *Journal of Mathematical Analysis and Applications*, 343:464–478.

[16] Ben Cheikh, Y. and Zaghouani, A. (2003). Some discrete d-orthogonal polynomial sets. *Journal of Computational and Applied Mathematics*, 156:253–263.

[17] Ben Cheikh, Y. and Zaghouani, A. (2007). d-orthogonality via generating functions. *Journal of Computational and Applied Mathematics*, 199:2–22.

[18] Ben Romdhane, N. (2007). d-orthogonal Faber polynomials. *Integral Transforms and Special Functions*, 18:663–677.

[19] Chaggara, H. and Gahami, A. (2024a). Generalized Dunkl-Appell polynomials. *Mediterranean Journal of Mathematics*, 21:188.

[20] Chaggara, H. and Gahami, A. (2024b). On d-orthogonal polynomials with Brenke type generating functions. *Journal of Mathematical Analysis and Applications*, 535:128123.

[21] Chaggara, H. and Gahami, A. (2025). Classification of 2-orthogonal polynomials with Brenke type generating functions. *Journal of Approximation Theory*, 306:106125.

[22] Chaggara, H. and Mbarki, R. (2018). On d-orthogonal polynomials of Sheffer type. *Journal of Difference Equations and Applications*, 24:1808–1829.

[23] Cheon, G. S. and Jung, J. H. (2016). The q-Sheffer sequences of a new type and associated orthogonal polynomials. *Linear Algebra and its Applications*, 491:171–186.

[24] Douak, K. (1996). The relation of the d-orthogonal polynomials to the Appell polynomials. *Journal of Computational and Applied Mathematics*, 70:279–295.

[25] Douak, K. and Maroni, P. (1997a). On d-orthogonal Tchebychev polynomials. i. *Applied Numerical Mathematics*, 24:23–53.

[26] Douak, K. and Maroni, P. (1997b). On d-orthogonal Tchebychev polynomials. ii. *Methods and Applications of Analysis*, 4:404–429.

[27] Johnson, W. P. (1996). Some applications of the q-exponential formula. *Discrete Mathematics*, 157:207–225.

[28] Koekoek, R., Lesky, P. A., and Swarttouw, R. F. (2010). *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics. Springer, Berlin.

[29] Lamiri, I. (2013). d-orthogonality of discrete q-Hermite type polynomials. *Journal of Approximation Theory*, 170:116–133.

[30] Lamiri, I. and Ouni, A. (2008a). d-orthogonality of Hermite type polynomials. *Applied Mathematics and Computation*, 202:24–43.

[31] Lamiri, I. and Ouni, A. (2008b). d-orthogonality of Humbert and Jacobi type polynomials. *Journal of Mathematical Analysis and Applications*, 341:24–51.

[32] Maroni, P. (1989). L'orthogonalité et les récurrences de polynômes d'ordre supérieur à deux. *Annales de la Faculté des sciences de Toulouse: Mathématiques*, 10:105–139.

[33] Meixner, J. (1934). Orthogonale polynomsysteme mit einer besonderen gestalt der erzeugenden funktion. *Journal of the London Mathematical Society*, 1:6–13.

[34] Sheffer, I. M. (1939). Some properties of polynomials of type zero. *Duke Mathematical Journal*, pages 590–622.

[35] Van Iseghem, J. (1987). Vector orthogonal relations. Vector qd-algorithm. *Journal of Computational and Applied Mathematics*, 19:141–150.

[36] Varma, S. (2019). Some new d-orthogonal polynomial sets of Sheffer type. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 68:913–922.

[37] Varma, S. and Taşdelen, F. (2012). On a different kind of d-orthogonal polynomials that generalize the Laguerre polynomials. *Math. Æterna*, 2:561–572.

[38] Zaghouani, A. (2005). Some basic d-orthogonal polynomial sets. *Georgian Mathematical Journal*, 12:583–593.