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Approximation by Sequence of Operators Involving Analytic Functions

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Abstract: In this contribution, we define a new operator sequence which contains analytic functions. Using approximation techniques found by Korovkin, some results are derived. Moreover, a generalization of this operator sequence called Kantorovich type generalization is introduced.

Keywords: Szasz operator; generalized Appell polynomials; analytic functions; Steklov function

1. Introduction

Recently there has been an enormous interest for the extension Szasz operators [1] that are closely related with the Poisson distribution. By the help of Appell and Sheffer polynomials, the celebrated generalization of Szasz operators were obtained by Jakimovski–Leviatan [2] and Ismail [3]. Lately, in view of umbral calculus [4], further developments have been represented in [5–9]. In these contributions, special functions such as orthogonal polynomials and d-orthogonal polynomials have been used frequently.

In the light of this information, a new sequence of operators yielding the generalization of Boas–Buck type polynomials [10]

$$R(\nu)\,\psi(xS(\nu)+\sigma(\nu)) = \sum_{j=0}^{\infty}\theta_j(x)\,\nu^j \tag{1}$$

is of the form

$$\mathcal{H}_n(f;x) = \frac{1}{R(1)\,\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\,f\left(\frac{j}{n}\right), \tag{2}$$

where R, ψ, S and σ have formal power series representations at the disc $|z| < L$ $(L > 1)$ with the following exception

$$S(\nu) = \sum_{j=0}^{\infty} s_j \nu^{j+1},\ (s_0 \neq 0) \text{ and } \sigma(\nu) = \sum_{j=0}^{\infty}\sigma_j\nu^{j+2}.$$

For the positivity and convergence problem of the sequence of operators (2), the following restrictions are crucial

(1) ψ is a real variable function with the range $(0,\infty)$,
(2) $\theta_j(x) \geq 0,\ j = 0,1,2,...,$
(3) $R(1) > 0$ and $S'(1) = 1$.

In the next section, convergence properties of the sequence of operators (2) will be discussed and, moreover, some estimations for approximation results by using a variety of mathematical instruments will be presented. In the last section, we will define the Kantorovich extension of (2) given by

$$\mathcal{H}_n^* (f;x) = n \frac{1}{R(1)\,\psi(nxS(1)+\sigma(1))} \sum_{j=0}^{\infty} \theta_j (nx) \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(\zeta)\, d\zeta, \tag{3}$$

and obtain some results related with this sequence of operators.

2. Some Results Related to the Sequence of Operators $\mathcal{H}_n$

First, we obtain some equalities which will be used in the sequel.

Lemma 1.

$$\begin{aligned}
\mathcal{H}_n (1;x) &= 1, \\
\mathcal{H}_n (\xi;x) &= \frac{\psi'(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))} x + \left(\frac{R'(1)}{R(1)} + \frac{\psi'(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))} \sigma'(1) \right) \frac{1}{n}, \\
\mathcal{H}_n \left(\xi^2;x\right) &= \frac{\psi''(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))} x^2 \\
&\quad + \left\{ \frac{\left[2R'(1) + \left(1+S''(1)\right) R(1)\right] \psi'(nxS(1)+\sigma(1))}{R(1)\,\psi(nxS(1)+\sigma(1))} \right. \\
&\quad \left. + \frac{2\sigma'(1)\,\psi''(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))} \right\} \frac{x}{n} \\
&\quad + \left\{ \frac{R''(1)+R'(1)}{R(1)} \right. \\
&\quad + \frac{\left[\left(2R'(1)+R(1)\right)\sigma'(1) + R(1)\,\sigma''(1)\right] \psi'(nxS(1)+\sigma(1))}{R(1)\,\psi(nxS(1)+\sigma(1))} \\
&\quad \left. + \frac{\left[\sigma'(1)\right]^2 \psi''(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))} \right\} \frac{1}{n^2}.
\end{aligned}$$

Proof. Taking $\nu = 1$ and $x \to nx$ in (1)

$$\sum_{j=0}^{\infty} \theta_j (nx) = R(1)\,\psi(nxS(1)+\sigma(1))$$

yields

$$\mathcal{H}_n (1;x) = \frac{1}{R(1)\,\psi(nxS(1)+\sigma(1))} \sum_{j=0}^{\infty} \theta_j (nx) = 1.$$

Applying the derivative operator with respect to ν leads to

$$\sum_{j=1}^{\infty} \theta_j (x)\, j\nu^{j-1} = R'(\nu)\,\psi(xS(\nu)+\sigma(\nu)) + R(\nu)\,\psi'(xS(\nu)+\sigma(\nu)) \left[xS'(\nu)+\sigma'(\nu)\right],$$

then a similar approach gives us

$$\sum_{j=0}^{\infty}\theta_j(nx)\,j = R'(1)\,\psi(nxS(1)+\sigma(1)) + R(1)\,\psi'(nxS(1)+\sigma(1))\left[nxS'(1)+\sigma'(1)\right]. \quad (4)$$

Thus

$$\begin{aligned}\mathcal{H}_n(\xi;x) &= \frac{1}{R(1)\,\psi(nxS(1)+\sigma(1))\,n}\sum_{j=0}^{\infty}\theta_j(nx)\,j\\ &= \frac{\psi'(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))}x+\left(\frac{R'(1)}{R(1)}+\frac{\psi'(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))}\sigma'(1)\right)\frac{1}{n}.\end{aligned}$$

Since the second derivative of (1) with respect to ν is of the following form

$$\begin{aligned}\sum_{j=2}^{\infty}\theta_j(x)\,j(j-1)\,\nu^{j-2} &= R''(\nu)\,\psi(xS(\nu)+\sigma(\nu)) + 2R'(\nu)\,\psi'(xS(\nu)+\sigma(\nu))\left[xS'(\nu)+\sigma'(\nu)\right]\\ &\quad + R(\nu)\,\psi''(xS(\nu)+\sigma(\nu))\left[xS'(\nu)+\sigma'(\nu)\right]^2\\ &\quad + R(\nu)\,\psi'(xS(\nu)+\sigma(\nu))\left[xS''(\nu)+\sigma''(\nu)\right],\end{aligned}$$

we conclude the below equality by using (4)

$$\begin{aligned}\sum_{j=0}^{\infty}\theta_j(nx)\,j^2 &= \left[R''(1)+R'(1)\right]\psi(nxS(1)+\sigma(1))\\ &\quad + \left[2R'(1)+R(1)\right]\psi'(nxS(1)+\sigma(1))\left[nxS'(1)+\sigma'(1)\right]\\ &\quad + R(1)\,\psi''(nxS(1)+\sigma(1))\left[nxS'(1)+\sigma'(1)\right]^2\\ &\quad + R(1)\,\psi'(nxS(1)+\sigma(1))\left[nxS''(1)+\sigma''(1)\right].\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{H}_n\left(\xi^2;x\right) &= \frac{1}{R(1)\,\psi(nxS(1)+\sigma(1))\,n^2}\sum_{j=0}^{\infty}\theta_j(nx)\,j^2\\ &= \frac{\psi''(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))}x^2+\left\{\frac{\left[2R'(1)+\left(1+S''(1)\right)R(1)\right]\psi'(nxS(1)+\sigma(1))}{R(1)\,\psi(nxS(1)+\sigma(1))}\right.\\ &\quad \left.+\frac{2\sigma'(1)\,\psi''(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))}\right\}\frac{x}{n}\\ &\quad +\left\{\frac{R''(1)+R'(1)}{R(1)}\right.\\ &\quad +\frac{\left[\left(2R'(1)+R(1)\right)\sigma'(1)+R(1)\,\sigma''(1)\right]\psi'(nxS(1)+\sigma(1))}{R(1)\,\psi(nxS(1)+\sigma(1))}\\ &\quad \left.+\frac{\left[\sigma'(1)\right]^2\psi''(nxS(1)+\sigma(1))}{\psi(nxS(1)+\sigma(1))}\right\}\frac{1}{n^2}.\end{aligned}$$

□

Lemma 2.

$$\begin{aligned}
\mathcal{H}_n\left(\xi - x; x\right) &= \left(\frac{\psi^{'}\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)} - 1\right)x + \left(\frac{R^{'}\left(1\right)}{R\left(1\right)} + \frac{\psi^{'}\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)}\sigma^{'}\left(1\right)\right)\frac{1}{n},\\
\mathcal{H}_n\left(\left(\xi - x\right)^2; x\right) &= \frac{\psi^{''}\left(nxS\left(1\right)+\sigma\left(1\right)\right) - 2\psi^{'}\left(nxS\left(1\right)+\sigma\left(1\right)\right) + \psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)}x^2\\
&+ \left\{\frac{\left[2R^{'}\left(1\right) + \left(1 + S^{''}\left(1\right) - 2\sigma^{'}\left(1\right)\right)R\left(1\right)\right]\psi^{'}\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{R\left(1\right)\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)}\right.\\
&\left. + \frac{2\sigma^{'}\left(1\right)\psi^{''}\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)} - 2\frac{R^{'}\left(1\right)}{R\left(1\right)}\right\}\frac{x}{n}\\
&+ \left\{\frac{R^{''}\left(1\right) + R^{'}\left(1\right)}{R\left(1\right)}\right.\\
&+ \frac{\left[\left(2R^{'}\left(1\right) + R\left(1\right)\right)\sigma^{'}\left(1\right) + R\left(1\right)\sigma^{''}\left(1\right)\right]\psi^{'}\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{R\left(1\right)\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)}\\
&\left. + \frac{\left[\sigma^{'}\left(1\right)\right]^2\psi^{''}\left(nxS\left(1\right)+\sigma\left(1\right)\right)}{\psi\left(nxS\left(1\right)+\sigma\left(1\right)\right)}\right\}\frac{1}{n^2}.
\end{aligned}$$

Proof. The claims hold in view of the following identities immediately

$$\mathcal{H}_n\left(\xi - x; x\right) = \mathcal{H}_n\left(\xi; x\right) - x\mathcal{H}_n\left(1; x\right),$$

$$\mathcal{H}_n\left(\left(\xi - x\right)^2; x\right) = \mathcal{H}_n\left(\xi^2; x\right) - 2x\mathcal{H}_n\left(\xi; x\right) + x^2\mathcal{H}_n\left(1; x\right).$$

□

Throughout the paper, the below assumptions on the function ψ

$$\lim_{y\to\infty}\frac{\psi^{''}\left(y\right)}{\psi\left(y\right)} = 1 \text{ and } \lim_{y\to\infty}\frac{\psi^{'}\left(y\right)}{\psi\left(y\right)} = 1 \tag{5}$$

will be considered. Now, we are on the verge of proving our main theorem.

Theorem 1. *Assume that f is continuous on $[0,\infty)$ and plus, this function also belongs to the class*

$$\Omega = \left\{f : \frac{f\left(x\right)}{1+x^2} \to A, \text{ when } x \to \infty\right\}.$$

Then, the uniform convergence of the operators sequence (2) is satisfied on the compact subsets of the $[0,\infty)$, i.e.,

$$\lim_{n\to\infty}\mathcal{H}_n\left(f; x\right) = f\left(x\right).$$

Proof. From Lemma 1, it is easy to find

$$\lim_{n\to\infty}\mathcal{H}_n\left(\xi^i; x\right) = x^i,\ i = 0, 1, 2.$$

This means that these convergences are provided uniformly on the compact subsets of the non-negative real axis. So, the proof is completed by using the universal Korovkin theorem [11]. □

In the following theorem, we suppose that f is uniform continuous on $[0,\infty)$.

Theorem 2.

$$|\mathcal{H}_n(f;x)-f(x)| \leq 2\omega\left(f;\sqrt{\mathcal{H}_n\left((\xi-x)^2;x\right)}\right),$$

where ω is the modulus of continuity of the function f [12] defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\leq\delta}} |f(x)-f(y)|.$$

Proof. Using triangle inequality and applying the well-known property of $\omega(f;\delta)$ leads us to

$$\begin{aligned} |\mathcal{H}_n(f;x)-f(x)| &= \left|\tfrac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\left(f\left(\tfrac{j}{n}\right)-f(x)\right)\right| \\ &\leq \tfrac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\left|f\left(\tfrac{j}{n}\right)-f(x)\right| \\ &\leq \tfrac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\left(1+\tfrac{1}{\delta}\left|\tfrac{j}{n}-x\right|\right)\omega(f;\delta) \\ &= \left(1+\tfrac{1}{\delta}\tfrac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\left|\tfrac{j}{n}-x\right|\right)\omega(f;\delta). \end{aligned} \tag{6}$$

For the infinite sum, the following result holds by the Cauchy–Schwarz inequality

$$\begin{aligned} \sum_{j=0}^{\infty}\theta_j(nx)\left|\tfrac{j}{n}-x\right| &= \sum_{j=0}^{\infty}\sqrt{\theta_j(nx)}\sqrt{\theta_j(nx)}\left|\tfrac{j}{n}-x\right| \\ &\leq \left[\sum_{j=0}^{\infty}\theta_j(nx)\right]^{1/2}\left[\sum_{j=0}^{\infty}\theta_j(nx)\left(\tfrac{j}{n}-x\right)^2\right]^{1/2} \\ &= R(1)\psi(nxS(1)+\sigma(1))\left[\mathcal{H}_n\left((\xi-x)^2;x\right)\right]^{1/2}. \end{aligned}$$

If we consider the last inequality in (6), this provides

$$|\mathcal{H}_n(f;x)-f(x)| \leq \left(1+\frac{1}{\delta}\left[\mathcal{H}_n\left((\xi-x)^2;x\right)\right]^{1/2}\right)\omega(f;\delta).$$

Here, by taking

$$\delta = \left[\mathcal{H}_n\left((\xi-x)^2;x\right)\right]^{1/2},$$

we get the desired result. □

Now, for $0<\rho\leq 1$ and $\eta_1,\eta_2\in[0,\infty)$, let us introduce the following class of functions

$$Lip_K^{(\rho)} = \left\{\varphi : |\varphi(\eta_1)-\varphi(\eta_2)| \leq K|\eta_1-\eta_2|^{\rho}\right\}.$$

Theorem 3. *Suppose that $\varphi\in Lip_K^{(\rho)}$, then*

$$|\mathcal{H}_n(\varphi;x)-\varphi(x)| \leq K\left[\mathcal{H}_n\left((\xi-x)^2;x\right)\right]^{\frac{\rho}{2}}.$$

Proof. Since $\varphi \in Lip_K^{(\rho)}$, we find

$$\begin{aligned} |\mathcal{H}_n(\varphi;x)-\varphi(x)| &= |\mathcal{H}_n(\varphi(\xi)-\varphi(x);x)| \\ &\leq \mathcal{H}_n(|\varphi(\xi)-\varphi(x)|;x) \\ &\leq K\mathcal{H}_n(|\xi-x|^{\rho};x). \end{aligned} \tag{7}$$

From (7), it becomes

$$\begin{aligned} \mathcal{H}_n(|\xi-x|^{\rho};x) &= \frac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\left|\frac{j}{n}-x\right|^{\rho} \\ &= \frac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\left[\theta_j(nx)\right]^{\frac{2-\rho}{2}}\left[\theta_j(nx)\right]^{\frac{\rho}{2}}\left|\frac{j}{n}-x\right|^{\rho} \\ &\leq \frac{1}{R(1)\psi(nxS(1)+\sigma(1))} \\ &\quad\times\left[R(1)\psi(nxS(1)+\sigma(1))\right]^{\frac{2-\rho}{2}}\left\{\frac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\right\}^{\frac{2-\rho}{2}} \\ &\quad\times\left[R(1)\psi(nxS(1)+\sigma(1))\right]^{\frac{\rho}{2}}\left\{\frac{1}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\left|\frac{j}{n}-x\right|^{2}\right\}^{\frac{\rho}{2}} \\ &= \left[\mathcal{H}_n(1;x)\right]^{\frac{2-\rho}{2}}\left[\mathcal{H}_n\left((\xi-x)^2;x\right)\right]^{\frac{\rho}{2}}, \end{aligned}$$

where we use the Hölder inequality. This proves the desired result. □

Next, we present a theorem related to the quantitative estimation of the sequence of operators (2). Let us first introduce Rasa's result and the second order Steklov function which will be used in the following theorem.

Let $z \in C^2[0,a]$ and $(L_n)_{n\geq 0}$ be a sequence of linear positive operators with the property $L_n(e_0;x)=e_0(x)$, $e_i(\xi)=\xi^i$, $i\in\{0,1,2\}$. Then, Rasa's result is known as

$$|L_n(z;x)-z(x)| \leq \left\|z'\right\|\sqrt{L_n\left((\xi-x)^2;x\right)}+\frac{1}{2}\left\|z''\right\|L_n\left((\xi-x)^2;x\right).$$

For $f\in C[a,b]$, the second order Steklov function of f is defined by

$$f_h(x) := \frac{1}{h}\int_{-h}^{h}\left(1-\frac{|t|}{h}\right)f(h;x+t)\,dt, \quad x\in[a,b],$$

where $f(h;.):[a-h,b+h]\to\mathbb{R}$, $h>0$, by

$$f(h;x)=\begin{cases} P_-(x) & ; \quad a-h\leq x\leq a \\ f(x) & ; \quad a\leq x\leq b \\ P_+(x) & ; \quad b<x\leq b+h \end{cases}$$

and P_-, P_+ are the best linear approximations to f on the indicated intervals.

Theorem 4. *Let φ be a continuous function on $[0,\infty)$. Then, it holds that*

$$|\mathcal{H}_n(\varphi;x)-\varphi(x)| \leq \frac{3}{2}\left(1+\frac{a}{2}+\frac{h^2}{2}\right)\omega_2(\varphi;h)+\frac{2h^2}{a}\|\varphi\|,$$

where ω_2 is the second order modulus of continuity of the function φ [12] defined by

$$\omega_2(\varphi;\delta) := \sup_{0<t\leq\delta} \|\varphi(.+2t) - 2\varphi(.+t) + \varphi(.)\|.$$

Proof. From some basic properties of the operators (2) and by simple algebraic computations, we reach

$$\mathcal{H}_n(\varphi;x) - \varphi(x) = \mathcal{H}_n(\varphi-\varphi_h;x) + \mathcal{H}_n(\varphi_h;x) - \varphi_h(x) + \varphi_h(x) - \varphi(x)$$

and so

$$\begin{array}{rcl} |\mathcal{H}_n(\varphi;x) - \varphi(x)| & \leq & \mathcal{H}_n(|\varphi-\varphi_h|;x) + |\mathcal{H}_n(\varphi_h;x) - \varphi_h(x)| + |\varphi_h(x) - \varphi(x)| \\ & \leq & 2\|\varphi-\varphi_h\| + |\mathcal{H}_n(\varphi_h;x) - \varphi_h(x)|, \end{array} \tag{8}$$

where φ_h is the second order Steklov function of φ [13]. It is widely know that $\varphi_h \in C^2[0,a]$. In view of this fact, we obtained the following expression by Rasa's result [14] and the Landau inequality

$$\begin{array}{rcl} |\mathcal{H}_n(\varphi_h;x) - \varphi_h(x)| & \leq & \left\|\varphi_h'\right\| \sqrt{\mathcal{H}_n\left((\xi-x)^2;x\right)} + \frac{1}{2}\left\|\varphi_h''\right\| \mathcal{H}_n\left((\xi-x)^2;x\right) \\ & \leq & \left(\frac{2}{a}\|\varphi_h\| + \frac{a}{2}\left\|\varphi_h''\right\|\right)\sqrt{\mathcal{H}_n\left((\xi-x)^2;x\right)} \\ & & +\frac{1}{2}\left\|\varphi_h''\right\| \mathcal{H}_n\left((\xi-x)^2;x\right) \\ & \leq & \left(\frac{2}{a}\|\varphi\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(\varphi;h)\right)\sqrt{\mathcal{H}_n\left((\xi-x)^2;x\right)} \\ & & +\frac{3}{4}\frac{1}{h^2}\omega_2(\varphi;h)\,\mathcal{H}_n\left((\xi-x)^2;x\right). \end{array} \tag{9}$$

According to Zhuk [13], there is a connection between the second order Steklov function and $\omega_2(\varphi;h)$ as follows

$$\|\varphi-\varphi_h\| \leq \frac{3}{4}\omega_2(\varphi;h).$$

Using this inequality and (9) in (8) give us

$$\begin{array}{rcl} |\mathcal{H}_n(\varphi;x) - \varphi(x)| & \leq & 2\|\varphi-\varphi_h\| + |\mathcal{H}_n(\varphi_h;x) - \varphi_h(x)| \\ & \leq & \frac{3}{2}\omega_2(\varphi;h) + \left(\frac{2}{a}\|\varphi\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(\varphi;h)\right)\sqrt{\mathcal{H}_n\left((\xi-x)^2;x\right)} \\ & & +\frac{3}{4}\frac{1}{h^2}\omega_2(\varphi;h)\,\mathcal{H}_n\left((\xi-x)^2;x\right). \end{array}$$

Finally, we get

$$\begin{array}{rcl} |\mathcal{H}_n(\varphi;x) - \varphi(x)| & \leq & \frac{3}{2}\omega_2(\varphi;h) + \left(\frac{2}{a}\|\varphi\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(\varphi;h)\right)h^2 \\ & & +\frac{3}{4}\frac{1}{h^2}\omega_2(\varphi;h)\,h^4 \\ & = & \frac{3}{2}\left(1+\frac{a}{2}+\frac{h^2}{2}\right)\omega_2(\varphi;h) + \frac{2h^2}{a}\|\varphi\|, \end{array}$$

where $h = \sqrt[4]{\mathcal{H}_n\left((\xi-x)^2;x\right)}$. □

3. A Further Extension of Sequence of Operators $\mathcal{H}_n$

In this section, some results related to the sequence of operators (3) are analyzed. First, let us introduce the following useful lemma.

Lemma 3.

$$\begin{aligned}\mathcal{H}_n^*(1;x) &= 1,\\ \mathcal{H}_n^*(\xi;x) &= \mathcal{H}_n(\xi;x)+\frac{1}{2n},\\ \mathcal{H}_n^*\left(\xi^2;x\right) &= \mathcal{H}_n\left(\xi^2;x\right)+\frac{1}{n}\mathcal{H}_n(\xi;x)+\frac{1}{3n^2},\end{aligned}$$

where $\mathcal{H}_n$ *is defined by* (2).

Proof. These assertions are easily acquired by considering the sequence of operators (3) and Lemma 1. □

A similar approach used in Lemma 2 leads us to the following result.

Remark 1.

$$\begin{aligned}\mathcal{H}_n^*(\xi-x;x) &= \mathcal{H}_n(\xi-x;x)+\frac{1}{2n},\\ \mathcal{H}_n^*\left((\xi-x)^2;x\right) &= \mathcal{H}_n\left((\xi-x)^2;x\right)+\frac{1}{n}\mathcal{H}_n(\xi-x;x)+\frac{1}{3n^2}.\end{aligned}$$

Theorem 5. *Assume that* f *is continuous on* $[0,\infty)$ *and belongs to the class*

$$\Omega=\left\{f:\frac{f(x)}{1+x^2}\to A,\ when\ x\to\infty\right\}.$$

Then, the uniform convergence of the operators sequence (3) *is satisfied on the compact subsets of the* $[0,\infty)$, *i.e.,*

$$\lim_{n\to\infty}\mathcal{H}_n^*(f;x)=f(x).$$

Proof. From Lemma 3, it is easy to find

$$\lim_{n\to\infty}\mathcal{H}_n^*\left(\xi^i;x\right)=x^i,\ i=0,1,2.$$

This means that these convergences are provided uniformly on the compact subsets of the non-negative real axis. So, the proof is completed by using the universal Korovkin theorem [11]. □

In the following theorem, we suppose that f is uniform continuous on $[0,\infty)$.

Theorem 6.

$$\left|\mathcal{H}_n^*(f;x)-f(x)\right|\le 2\omega\left(f;\sqrt{\mathcal{H}_n^*\left((\xi-x)^2;x\right)}\right),$$

where ω *is the modulus of continuity [12].*

Proof. Using the triangle inequality and applying the well-known property of $\omega(f;\delta)$ leads us to

$$\begin{aligned}\left|\mathcal{H}_n^*(f;x)-f(x)\right| &\le \frac{n}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left|f(\zeta)-f(x)\right|d\zeta\\ &\le \frac{n}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(1+\tfrac{1}{\delta}|\zeta-x|\right)\omega(f;\delta)\,d\zeta \qquad (10)\\ &= \left(1+\tfrac{1}{\delta}\frac{n}{R(1)\psi(nxS(1)+\sigma(1))}\sum_{j=0}^{\infty}\theta_j(nx)\int_{\frac{j}{n}}^{\frac{j+1}{n}}|\zeta-x|\,d\zeta\right)\omega(f;\delta).\end{aligned}$$

For the infinite sum, the following result holds by the Cauchy–Schwarz inequality

$$\begin{aligned}\sum_{j=0}^{\infty}\theta_j(nx)\int_{\frac{j}{n}}^{\frac{j+1}{n}}|\zeta-x|\,d\zeta &\leq \frac{1}{\sqrt{n}}\sum_{j=0}^{\infty}\theta_j(nx)\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}(\zeta-x)^2\,d\zeta\right)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{n}}\sqrt{R(1)\,\psi(nxS(1)+\sigma(1))}\sqrt{\mathcal{H}_n(1;x)}\\ &\quad\times\sqrt{\frac{R(1)\,\psi(nxS(1)+\sigma(1))}{n}}\sqrt{\mathcal{H}_n^*\left((\xi-x)^2;x\right)}.\end{aligned}$$

If we consider the last inequality in (10), this provides

$$|\mathcal{H}_n^*(f;x)-f(x)|\leq\left(1+\frac{1}{\delta}\sqrt{\mathcal{H}_n^*\left((\xi-x)^2;x\right)}\right)\omega(f;\delta).$$

Here, by taking

$$\delta=\sqrt{\mathcal{H}_n^*\left((\xi-x)^2;x\right)},$$

we get the following desired result

$$|\mathcal{H}_n^*(f;x)-f(x)|\leq 2\omega\left(f;\sqrt{\mathcal{H}_n^*\left((\xi-x)^2;x\right)}\right).$$

□

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