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On Constant Slope Surfaces by Means of the Rotation Minimizing Frame in $\mathbb{E}^n$

Sevda Arslan * **and Yusuf Yaylı**

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ABSTRACT

In this paper, we give the characterizations of constant slope surfaces by means of the rotation minimizing frame (RMF). Also, constant slope surfaces are generalized to n-dimensional space E^n. Finally, some related examples are given.

Keywords: *Constant angle surface, spiral curve, Frenet frame, rotation minimizing frame, Darboux frame, slope surface, constant slope surface.*

AMS Subject Classification (2020): *Primary: 53A04 ; Secondary: 53A05.*

1. Introduction

The Frenet frame is an orthonormal frame which can be defined for curves in E^n. Rotation minimizing frames are an alternative to the Frenet. The Rotation minimizing frame has many applications in biology and computer graphics. For example, the RMF can be used to predict the shape of DNA sequences or to control virtual cameras in computer graphics, Mechanical and Machine Design. For studies on RMF, you can see articles [1] and [2].

Generalized constant slope surfaces are used in various fields such as engineering and mathematical modeling. The basic feature of these surfaces is that they have a certain angle of inclination. This feature is used in many areas. They are used in Geographic Information Systems (GIS) and Map Drawing to analyze the slope of a mountain or a hill, providing information about landforms and slopes, to simulate terrain models. It is used for slope calculations of a road or waterway project and for land analysis in fields such as civil engineering and environmental engineering. Generalized constant slope surfaces are also used to model the interaction of fluids with a surface in simulations such as fluid dynamics and fluid mechanics. This is particularly useful for understanding how air and water flow behave on sloped surfaces. Additionally, the fixed tilt allows optimizing the efficiency of solar panels. Furthermore, when robots need to move on inclined surfaces, generalized constant slope surfaces are important for planning and simulating the movement. Especially autonomous vehicles or mobile robots need such surfaces to analyze and safely move on inclined surfaces. Besides, constant slope surfaces are used on surfaces where machines work, especially to ensure that machines move smoothly on a surface. For example, conveyor belts or components of machines can work on such surfaces.

In short, generalized constant slope surfaces play an important role in real-world problems, various engineering, architectural, physical and geographic analyses. The slope of these surfaces is a critical parameter for understanding and modeling the events that occur on them.

In this paper; firstly, we considered the rotation minimizing frame in 3-dimensional Euclidean space and obtained constant slope surfaces with this frame. Then, we obtained constant slope surfaces in E^4 and E^5 using a similar method. Finally, we generalized this to n-dimensional Euclidean space and supported this with examples.

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2. Preliminaries

2.1. Frenet Frame

Let α: $\mathcal{I} \subset R \to R^3$ be a curve in 3-dimensional Euclidean space. The vectors forming the Frenet frame $\{T, N, B\}$ along the α curve can be defined as follows:

- T is the unit tangent vector of the curve, indicating the direction of motion.
- N is the unit normal vectors, the derivative of T with respect to the arc length parameter of the curve divided by its length.
- B is the vector product of two unit vectors, T and N.

T=$\frac{\alpha'}{\|\alpha'\|}$ N=$\frac{T'}{\|T'\|}$ B=T x N

1) If α is a curve with unit speed (parameterized by arclength function), that is, if $\|\alpha'\| = 1$, Frenet vectors are as follows;

$$\mathbf{T} = \alpha', \quad \mathbf{N} = \frac{\alpha''}{\|\alpha''\|} \quad \text{and} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

Here $\kappa = \|\mathbf{T}'\|$ is curvature, $\tau = - < \mathbf{B}', \mathbf{N} >$ torsion.

Proposition 2.1. *For the curve α to be a straight line, the necessary and sufficient condition is that the curvature of the curve α is zero.($\kappa = 0$)*

Proposition 2.2. *Let α be a unit speed curve. For the α curve to be a planar curve, the necessary and sufficient condition is that the torsion of the α curve is zero. ($\tau = 0$)*

2) If $\alpha(s)$ is not a unit speed curve, Frenet vectors can also be written as;

$$\mathbf{T} = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \qquad \mathbf{N} = \mathbf{B}\mathbf{x}\mathbf{T}, \qquad \mathbf{B} = \frac{\alpha'(s) \ \mathbf{x} \ \alpha''(s)}{\|\alpha'(s) \ \mathbf{x} \ \alpha''(s)\|}.$$

$$\kappa = \frac{\|\alpha' \mathbf{x} \alpha''\|}{\|\alpha'\|^3} \qquad \text{and} \qquad \tau = \frac{det(\alpha', \alpha'', \alpha''')}{\|\alpha' \ \mathbf{x} \ \alpha''\|^2}.$$

Frenet vectors are assumed to rotate around an axis along the curve depending on the parameter. This axis is called the Darboux (sudden rotation) axis of the curve, and the vector that gives the direction and orientation of this axis is called the Darboux vector (angular velocity vector) and is indicated by $\mathbf{W}$. This vector is given by the relation $\mathbf{W}$=$\tau\mathbf{T}$ + $\kappa\mathbf{B}$ and its norm is calculated as: $\|\mathbf{W}\|$=$\sqrt{\tau^2 + \kappa^2}$; the roof with minimum angular velocity is called rotation minimizing frame.

2.2. Rotation Minimizing Frame (Bishop Frame)

If there is a curve such that α: $\mathcal{I} \to \mathcal{R}^3$ and if the derivative of the normal vectors of this curve is in the direction of the tangent vector of the curve, then this frame is called an RMF (Bishop frame). The components

of the Bishop frame formed by rotating the Frenet frame of an α curve given with the arc parameter s in E^3 by θ about $\mathbf{T}$ are $\{\mathbf{T}, \mathbf{N_1}, \mathbf{N_2}, k_1, k_2\}$ and the Bishop formulas of the α curve with respect to the arc parameter s are

$$\mathbf{T}' = k_1\mathbf{N_1} + k_2\mathbf{N_2}, \quad \mathbf{N_1}' = -k_1\mathbf{T} \quad \text{and} \quad \mathbf{N_2}' = -k_2\mathbf{T}.$$

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N_1} \\ \mathbf{N_2} \end{bmatrix}' = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N_1} \\ \mathbf{N_2} \end{bmatrix}. \tag{2.1}$$

There are relations between the Frenet and Bishop components of the α curve

$$\begin{aligned} \mathbf{T} &= \alpha', \\ \mathbf{N} &= \cos\theta\mathbf{N_1} + \sin\theta\mathbf{N_2}, \\ \mathbf{B} &= -\sin\theta\mathbf{N_1} + \cos\theta\mathbf{N_2} \end{aligned} \tag{2.2}$$

and

$$\tau(s) = \theta'(s) \quad \text{and} \quad \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)}. \tag{2.3}$$

From here

$$k_1(s) = \kappa(s)\cos\theta(s) \qquad \text{and} \qquad k_2(s) = \kappa(s)\sin\theta(s).$$

Then, the Darboux vector of this RMF (Bishop frame) is given by

$$\overline{\mathbf{W}} = -k_2\mathbf{N_1} + k_1\mathbf{N_2}.$$

It can be easily seen that $\overline{\mathbf{W}}$=$\kappa\mathbf{B}$. The norm of $\overline{\mathbf{W}}$ is calculated as: $\|\overline{\mathbf{W}}\|$=$\sqrt{k_1^2 + k_2^2}$ or it can be expressed as: $\|\overline{\mathbf{W}}\|$=$\kappa$. So, $\|\overline{\mathbf{W}}\| < \|\mathbf{W}\|$. For this reason, these frames are called rotation minimizing frame (RMF). Because the angular velocity is minimum in this frame.

In $n-$dimensions, the RMF is $\{\mathbf{T(s)}, \mathbf{N_1(s)}, \mathbf{N_2(s)}, ..., \mathbf{N_{n-1}(s)}\}$ and as follows:

$$\begin{bmatrix} \mathbf{T(s)} \\ \mathbf{N_1(s)} \\ \mathbf{N_2(s)} \\ . \\ . \\ . \\ \mathbf{N_{n-1}(s)} \end{bmatrix}' = \begin{bmatrix} 0 & k_1(s) & k_2(s) & ... & k_{n-1}(s) \\ -k_1(s) & 0 & 0 & ... & ... \\ -k_2(s) & 0 & 0 & ... & ... \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ -k_{n-1}(s) & 0 & 0 & ... & ... \end{bmatrix} \begin{bmatrix} \mathbf{T(s)} \\ \mathbf{N_1(s)} \\ \mathbf{N_2(s)} \\ . \\ . \\ . \\ \mathbf{N_{n-1}(s)} \end{bmatrix} \tag{2.4}$$

where the function $k_i(s)$, $i \in \{1, 2, 3, 4, ..., n-1\}$, denote the curvatures of the curve and all $k_i(s)$ are positive.

3. Constant Slope Surfaces with RMF

The surface is called a constant slope surface if the angle between the posision vector and the normal at any point on a surface is constant. All curves on constant slope surfaces are constant slope curves. We can obtain

constant-slope surfaces by rotating the spiral curve. We can also obtain surfaces of revolution using the curve's RMF. Thus, we can see the relationship between the RMF and constant slope surfaces. Based on this, we can see that constant slope hypersurfaces and constant ratio hypersurfaces can be obtained with the RMF of constant slope E^n .

Let us begin by examining the spiral curve. This curve has the property that the angle θ between its tangent and the radial direction is constant at every point. For this reason, it is generally known as an equiangular spiral. In polar coordinates (p, φ), it is defined by the equation $p = a.e^{b\psi}$, where $b = \cot\theta$, and a and b are positive real constants. It has also been proven that in planar coordinates,

$$x(t) = ae^{t\cos\theta}\cos t$$

$$y(t) = ae^{t\cos\theta}\sin t$$

where the origin at $t = 0$ is (a,0).

3.1. Characterization of Constant Slope Surfaces

Let $r : S \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface which is given by

$$r(u, v) = u\sin\theta(\cos\xi f(v) + \sin\xi.f(v) \text{ x } f'(v))$$

where θ is a fixed angle different from zero. ξ=$\xi(u)$=$\cot\theta.\log u$ and f is the unit velocity curve on S^2.

It is clear that the position vector is perpendicular to the surface at S^2. In this case, the angle is zero. Now let us prove that the surface equation specifies a constant slope surface in E^3. Firstly, the unit normal of S is

$$\mathbf{U} = \cos(\xi - \theta)f(v) + \sin(\xi - \theta)f(v) \text{ x } f'(v).$$

Then, the angle between **U** and the position vector is constant.

Here, we will obtain the constant slope surface given in point [1] on any curve using RMF.

3.1.1. Expressing constant slope surfaces with RMF

Theorem 3.1. *: Let $\{T, N_1, N_2\}$ be an RMF on any curve α in E^3. The surface given by*

$$\Phi(u, v) = u\sin\theta(\cos\xi N_1(v) + \sin\xi N_2(v))$$

is a constant slope surface, where $\xi(u) = \xi = \cot\theta.\ln u$.

Proof. : Let $\{T, N_1, N_2\}$ be an RMF on any curve α in E^3 and the surface Φ be defined as:

$$\Phi(u, v) = u\sin\theta(\cos\xi N_1(v) + \sin\xi N_2(v)).$$

First,let's find the unit normal vector of the surface Φ. If we take the derivatives of Φ with the respect to u and v, we get

$$\Phi_u = \sin(\theta - \xi)N_1(v) + \cos(\theta - \xi)N_2(v),$$

$$\Phi_v = u\sin\theta(\cos\xi N_1'(v) + \sin\xi N_2'(v)).$$

By using (2.1) in above equalities, the normal vector **U** is obtained as:

$$\mathbf{U} = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|} = \cos(\xi - \theta)N_1(v) + \sin(\xi - \theta)N_2(v).$$

On the other hand,the unit vector in the direction of the position vector of the surface Φ is found as below:

$$\frac{\Phi}{\|\Phi\|} = \cos\xi N_1(v) + \sin\xi N_2(v).$$

Thus

$$\left\langle \mathbf{U}, \frac{\Phi}{\|\Phi\|} \right\rangle = \cos\xi\cos(\xi-\theta)N_1 \times N_1 + \cos\xi\sin(\xi-\theta)N_1 \times N_2$$

$$+\sin\xi\cos(\xi-\theta)N_1 \times N_2 + \sin\xi\sin(\xi-\theta)N_2 \times N_2.$$

Since the angle between the position vector of the surface and its unit normal vector is constant, the surface Φ is a constant slope surface.

□

Corollary 3.1. *Let $f(v) \subset S^2$ be a unit speed curve; the $\{f, f', f \times f'\}$ Darboux frame on $f(v)$ is an RMF.*

$$\begin{bmatrix} f \\ f' \\ f \times f' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} f \\ f' \\ f \times f' \end{bmatrix}$$

If $f' = T$, $f = N_1$ and $f \times f' = N_2$, then

$$\Phi(u,v) = u\sin\theta(\cos\xi f(v) + \sin\xi f(v)\ x\ f'(v))$$

is obtained. Therefore, this result shows that a constant slope surface given in result [1] can be obtained on any curve using RMF.

Example 3.1. Let us write a constant slope surface for the helix curve $\alpha(v) = (\cos v, \sin v, v)$. If we take the derivative of α, we have $\alpha'(v) = (-\sin v, \cos v, 1)$ and $\|\alpha'\| = \sqrt{2}$. Hence the Frenet vectors and curvatures are computed as, respectively:

$$\begin{aligned} T &= (\tfrac{-1}{\sqrt{2}}\sin v, \tfrac{1}{\sqrt{2}}\cos v, \tfrac{1}{\sqrt{2}}), \\ N &= (-\cos v, -\sin v, 0), \\ B &= (\tfrac{1}{\sqrt{2}}\sin v, -\tfrac{1}{\sqrt{2}}\cos v, \tfrac{1}{\sqrt{2}}) \end{aligned}$$

and

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{1}{2} \text{ and } \tau = \frac{det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2} = \frac{1}{2}$$

By using the relations Frenet frame and Bishop frame, we get

$$\theta(v) = \int \tau dv = \frac{1}{2}v.$$

Hence

$$N_1 = \cos\theta(v)N - \sin\theta(v)B = (-\cos\frac{v}{2}\cos v - \frac{1}{\sqrt{2}}\sin\frac{v}{2}\sin v, -\cos\frac{v}{2}\sin v + \frac{1}{\sqrt{2}}\sin\frac{v}{2}\cos v, -\frac{1}{\sqrt{2}}\sin\frac{v}{2}),$$

$$N_2 = \sin\theta(v)N + \cos\theta(v)B = (-\sin\frac{v}{2}\cos v + \frac{1}{\sqrt{2}}\cos\frac{v}{2}\sin v, -\sin\frac{v}{2}\sin v - \frac{1}{\sqrt{2}}\cos\frac{v}{2}\cos v, -\frac{1}{\sqrt{2}}\cos\frac{v}{2}).$$

Using these N_1 and N_2, we can write the surface with constant slope

$$\Phi(u,v) = u\sin\theta(\cos\xi N_1(v) + \sin\xi N_2(v)).$$

Example 3.2. Let us write a constant slope surface for the curve $\alpha(v) = (24\cos\frac{v}{25}, 24\sin\frac{v}{25}, \frac{7v}{25})$. Hence the Frenet vectors and curvatures are computed as, respectively:

$$\alpha'(v) = (\frac{-24}{25}\sin\frac{v}{25}, \frac{24}{25}\cos\frac{v}{25}, \frac{7}{25}),$$

$$\alpha''(v) = (\frac{-24}{625}\cos\frac{v}{25}, \frac{-24}{625}\sin\frac{v}{25}, 0).$$

$$\|\alpha'(v)\| = 1 \text{ a unit speed curve, } \|\alpha''(v)\| = \frac{24}{625}.$$

$$\begin{aligned} T &= (\tfrac{-24}{25}\sin\tfrac{v}{25}, \tfrac{24}{25}\cos\tfrac{v}{25}, \tfrac{7}{25}), \\ N &= (-\cos\tfrac{v}{25}, -\sin\tfrac{v}{25}, 0), \\ B &= (\tfrac{7}{25}\sin\tfrac{v}{25}, \tfrac{-7}{25}\cos\tfrac{v}{25}, \tfrac{24}{25}), \end{aligned}$$

$$\kappa = \frac{\|\alpha'\times\alpha''\|}{\|\alpha'\|^3} = \frac{24}{625} \quad \text{and} \quad \tau = \frac{7}{625}.$$

Here,

$$\theta(v) = \int \tau dv = \frac{7}{625}v.$$

By using the relations Frenet frame and Bishop frame, we get

$$N_1 = \cos\theta(v)N - \sin\theta(v)B$$

$$= (-\cos\frac{7v}{625}\cos\frac{v}{25} - \frac{7}{25}\sin\frac{7v}{625}\sin\frac{v}{25}, -\cos\frac{7v}{625}\sin\frac{v}{25} + \frac{7}{25}\sin\frac{7v}{625}\cos\frac{v}{25}, -\frac{24}{25}\sin\frac{7v}{625}),$$

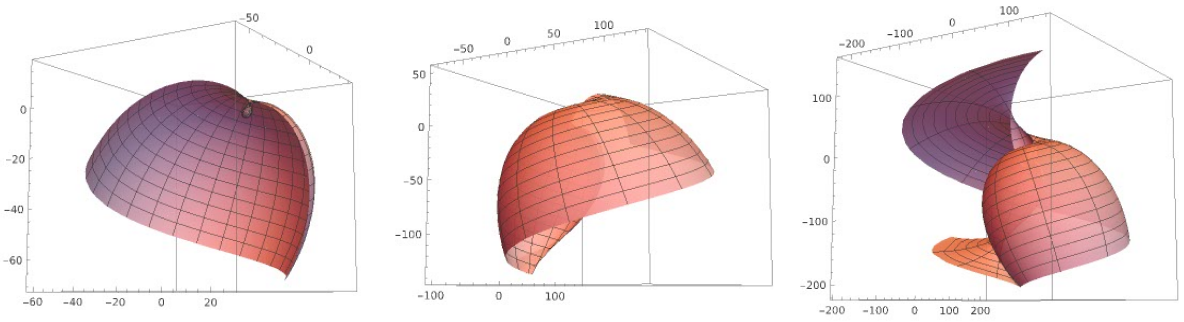
$$N_2 = \sin\theta(v)N + \cos\theta(v)B$$

$$= (-\sin\frac{7v}{625}\cos\frac{v}{25} + \frac{7}{25}\cos\frac{7v}{625}\sin\frac{v}{25}, -\sin\frac{7v}{625}\sin\frac{v}{25} - \frac{7}{25}\cos\frac{7v}{625}\cos\frac{v}{25}, \frac{24}{25}\cos\frac{7v}{625}).$$

Using these N_1 and N_2, we can write the surface with constant slope

$$\Phi(u,v) = u\sin\theta(\cos\xi N_1(v) + \sin\xi N_2(v)).$$

In this example, if we specifically choose $\theta = \pi/4$, the visual representation of our constant slope surface will look like this:



3.2. Constant slope surfaces on E^4 with RMF

Choosing any curve as α and then applying a frame on it, we can define 3 different types constant slope surfaces. And let's express this with the following theorem.

Theorem 3.2. *: Let $\{T, N_1, N_2, N_3\}$ be an RMF on any curve α in E^4. With this RMF, 3 types of constant slope surfaces can be written.*

• *1. type:*

$$\Phi_1(u,v,t_1) = u\sin\theta[\cos\xi N_1(v) + \sin\xi(f_1(t_1)N_2(v) + f_2(t_1)N_3(v))]$$

denotes a constant slope surface. Here, $f_1^2 + f_2^2 = 1$.

The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_1(v) + \sin(\xi-\theta)\big(f_1(t_1)N_2(v) + f_2(t)N_3(v)\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *2. type:*

$$\Phi_2(u,v,t_1) = u\sin\theta[\cos\xi N_2(v) + \sin\xi(f_1(t_1)N_1(v) + f_2(t_1)N_3(v))]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_2(v) + \sin(\xi-\theta)\big(f_1(t_1)N_1(v) + f_2(t)N_3(v)\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *3. type:*

$$\Phi_3(u,v,t_1) = u\sin\theta[\cos\xi N_3(v) + \sin\xi(f_1(t_1)N_1(v) + f_2(t_1)N_2(v))]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_3(v) + \sin(\xi-\theta)\big(f_1(t_1)N_1(v) + f_2(t)N_2(v)\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta$$

3.3. Constant slope surfaces on E^5 with RMF

Any curve α and then using elements of an RMF on α, we can define 4 different types constant slope surfaces in E^5. And let's express this with the following theorem.

Theorem 3.3. *: Let $\{T, N_1, N_2, N_3, N_4\}$ be an RMF on any curve α in E^5. 4 types of constant slope surfaces can be written using this RMF.*

• *1. type:*

$$\Phi_1(u,v,t_1,t_2) = u\sin\theta\big[\cos\xi N_1(v) + \sin\xi(f_1(t_1,t_2)N_2(v) + f_2(t_1,t_2)N_3(v) + f_3(t_1,t_2)N_4(v))\big]$$

denotes a constant slope surface. Here, $f_1^2 + f_2^2 + f_3^2 = 1$.
The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_1(v) + \sin(\xi-\theta)\big(f_1(t_1,t_2)N_2(v) + f_2(t_1,t_2)N_3(v) + f_3(t_1,t_2)N_4(v)\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *2. type:*

$$\Phi_2(u,v,t_1,t_2) = u\sin\theta\big[\cos\xi N_2(v) + \sin\xi(f_1(t_1,t_2)N_1(v) + f_2(t_1,t_2)N_3(v) + f_3(t_1,t_2)N_4(v))\big]$$

denotes a constant slope surface. Here, $f_1^2 + f_2^2 + f_3^2 = 1$. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_2(v) + \sin(\xi-\theta).\big(f_1(t_1,t_2)N_1(v) + f_2(t_1,t_2)N_3(v) + f_3(t_1,t_2)N_4(v))\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *3. type:*

$$\Phi_3(u,v,t_1,t_2) = u\sin\theta\big[\cos\xi N_3(v) + \sin\xi(f_1(t_1,t_2)N_1(v) + f_2(t_1,t_2)N_2(v) + f_3(t_1,t_2)N_4(v))\big]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_3(v) + \sin(\xi-\theta)\big(f_1(t_1,t_2)N_1(v) + f_2(t_1,t_2)N_2(v) + f_3(t_1,t_2)N_4(v))\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *4. type:*

$$\Phi_4(u,v,t_1,t_2) = u\sin\theta\big[\cos\xi N_4(v) + \sin\xi(f_1(t_1,t_2)N_1(v) + f_2(t_1,t_2)N_2(v) + f_3(t_1,t_2)N_3(v))\big].$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_4(v) + \sin(\xi-\theta)\big(f_1(t_1,t_2)N_1(v) + f_2(t_1,t_2)N_2(v) + f_3(t_1,t_2)N_3(v))\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

3.4. Constant slope surfaces on E^6 with RMF

Similarly, using elements of an RMF on any curve α in E^6 , we can define 5 types constant slope surfaces in E^5. And let's express this with the following theorem.

Theorem 3.4. *: Let $\{T, N_1, N_2, N_3, N_4, N_5\}$ be an RMF on any curve α in E^6. 5 types of constant slope surfaces can be written using this RMF.*

• *1. type:*

$$\Phi_1(u,v,t_1,t_2,t_3) = u\sin\theta\big[\cos\xi N_1(v) + \sin\xi(f_1(t_1,t_2,t_3)N_2(v) + f_2(t_1,t_2,t_3)N_3(v) + f_3(t_1,t_2,t_3)N_4(v) + f_4(t_1,t_2,t_3)N_5(v))\big]$$

denotes a constant slope surface. Here, $f_1^2 + f_2^2 + f_3^2 + f_4^2 = 1$.
The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_1(v) + \sin(\xi-\theta)\big(f_1(t_1,t_2,t_3)N_2(v) + f_2(t_1,t_2,t_3)N_3(v) + f_3(t_1,t_2,t_3)N_4(v) + f_4(t_1,t_2,t_3)N_5(v)\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *2. type:*

$$\Phi_2(u,v,t_1,t_2,t_3) = u\sin\theta\big[\cos\xi N_2(v) + \sin\xi(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_3 + f_3(t_1,t_2,t_3)N_4 + f_4(t_1,t_2,t_3)N_5)\big]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_2 + \sin(\xi-\theta)\big(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_3 + f_3(t_1,t_2,t_3)N_4 + f_4(t_1,t_2,t_3)N_5\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *3. type:*

$$\Phi_3(u,v,t_1,t_2,t_3) = u\sin\theta\big[\cos\xi N_3 + \sin\xi(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_2 + f_3(t_1,t_2,t_3)N_4 + f_4(t_1,t_2,t_3)N_5)\big]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_3 + \sin(\xi-\theta)\big(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_2 + f_3(t_1,t_2,t_3)N_4 + f_4(t_1,t_2,t_3)N_5\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *4. type:*

$$\Phi_4(u,v,t_1,t_2,t_3) = u\sin\theta\big[\cos\xi N_4 + \sin\xi(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_2 + f_3(t_1,t_2,t_3)N_3 + f_4(t_1,t_2,t_3)N_5)\big]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_4 + \sin(\xi-\theta)\big(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_2 + f_3(t_1,t_2,t_3)N_3 + f_4(t_1,t_2,t_3)N_5\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

• *5. type:*

$$\Phi_5(u,v,t_1,t_2,t_3) = u\sin\theta\big[\cos\xi N_5 + \sin\xi(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_2 + f_3(t_1,t_2,t_3)N_3 + f_4(t_1,t_2,t_3)N_4)\big]$$

denotes a constant slope surface. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_5 + \sin(\xi-\theta)\big(f_1(t_1,t_2,t_3)N_1 + f_2(t_1,t_2,t_3)N_2 + f_3(t_1,t_2,t_3)N_3 + f_4(t_1,t_2,t_3)N_4\big).$$

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta.$$

3.5. Constant slope surfaces on E^n with RMF

Now, let us characterize in general $(n-1)$ types constant slope surfaces with RMF $\{T, N_1, N_2, ..., N_{n-1}\}$ in E^n.

Theorem 3.5. *: Let $\{T, N_1, N_2, ..., N_{n-1}\}$ be an RMF over any curve α in the space E^n for $n > 3$. With this RMF, $n-1$ types of constant slope surfaces can be written.*

• *1. type:*

$$\Phi_1(u, v, t_1, ..., t_{n-3}) = usin\theta\Big[\cos\xi N_1(v) + \sin\xi(f_1(t_1, t_2, ..., t_{n-3})N_2(v) + ...$$

$$+f_{n-2}(t_1, t_2, ..., t_{n-3})N_{n-1}(v))\Big]$$

denotes a constant slope surface. Here, $f_1^2 + f_2^2 + ... + f_{n-2}^2 = 1$. The normal of the surface is

$$\boldsymbol{U} = \cos(\xi-\theta)N_1(v) + \sin(\xi-\theta)\big(f_1(t_1, t_2, ..., t_{n-3})N_2(v) + ... + f_{n-2}(t_1, t_2, ..., t_{n-3})N_{n-1}(v)\big)$$

and

$$\left\langle \frac{\Phi}{\|\Phi\|}, \boldsymbol{U} \right\rangle = \cos\theta \quad \textit{is constant.}$$

• *2. type:*

$$\Phi_2(u, v, t_1, ..., t_{n-3}) = u\sin\theta\Big[\cos\xi N_2(v) + \sin\xi(f_1(t_1, t_2, ..., t_{n-3})(N_1(v) + ... + f_{n-2}(t_1, t_2, ..., t_{n-3})N_{n-1}(v))\Big].$$

.
.
.

• $n-1$. *type:*

$$\Phi_{n-1}(u, v, t_1, ..., t_{n-3}) = u\sin\theta\Big[\cos\xi N_{n-1}(v) + \sin\xi(f_1(t_1, t_2, ..., t_{n-3})N_1(v) + ... + f_{n-2}(t_1, t_2, ..., t_{n-3})N_{n-2}(v))\Big].$$

From this general case, if we choose n=4, we see that is obtained special case Teorem 3.2. Or, if we choose n=5, we see that is obtained Teorem 3.3.

In summary, we have shown that a constant slope surface can be obtained using the RMF defined for any given curve.

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