

Jost Solutions and Scattering Data for Impulsive Klein-Gordon S-Wave Equations

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ABSTRACT

In this study, we consider the Klein–Gordon s-wave equation defined on the half line under impulsive condition and construct the corresponding differential operator. We then introduce the transfer matrices associated with the impulsive Klein–Gordon s-wave operator on the half line and employ them to analyze the scattering function and its analytic structure through an alternative operator-theoretic method. This approach enables a new characterization of the scattering behavior and establishes connections between the analytic properties of the transfer matrices and the spectral features of the operator. In the final part of the study, we construct the resolvent operator and examine its analytic structure.

Keywords: Klein-Gordon equation, scattering function, impulsive condition, eigenvalue

2020 Mathematics Subject Classification: 47A40, 34L15, 34L25

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Research Article; Received: October 31, 2025; **Accepted:** January 13, 2026

Cite this article as: Erdal, İ. (2026). Jost solutions and scattering data for impulsive Klein-Gordon s-wave equations. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 75 (3), 358-373.

1 Introduction

Spectral and scattering theory form the mathematical foundation of the analysis of linear wave and quantum systems. Spectral theory characterizes the decomposition of linear operators into discrete and continuous spectral components and provides the tools needed to study stability, long-time dynamics, and resonance phenomena; scattering theory, in turn, describes how incoming waves are transformed by an interaction region and encodes this transformation in scattering data (scattering function, scattering matrix, reflection/transmission amplitudes, resonances). These two theories are intimately related: spectral singularities, resonances and eigenvalues manifest both as special analytic features of the resolvent and as zeros of the scattering function, and many inverse problems reduce to recovering operator coefficients from scattering or spectral information. Comprehensive studies of spectral and scattering theory for Sturm–Liouville, Schrödinger, and even other types of operators can be found in the literature, and they form the foundation for the present work [3, 14, 15, 17, 20].

From both a theoretical and applied view, problems with impulsive or interface conditions are of particular interest. Impulsive models, where the field or its derivative undergoes instantaneous jumps at specified spatial locations, arise naturally when idealizing thin layers, abrupt material changes, point interactions, or fast

localized forces. Mathematically, impulsive conditions convert a single-domain differential problem into a piecewise-smooth problem coupled through algebraic transmission conditions at the interfaces. These discontinuities alter the operator's domain and consequently produce new spectral and scattering phenomena, requiring adaptations of classical spectral and scattering techniques. The theory of impulsive equations has been greatly developed in the context of differential equations [2, 13, 21, 22]. From the points of spectral and scattering theory, various papers can be highly instructive for readers, as they investigate the spectral properties of different forms of equations with impulsive conditions [4, 9, 10, 16, 24, 25, 26].

The Klein–Gordon s wave equation occupies a central position among relativistic wave equations: it models scalar quantum fields, relativistic particle propagation, and various wave phenomena in dispersive media. Beyond quantum field theory, Klein–Gordon–type equations appear in nonlinear optics, material science, seismology, and models of wave propagation in structured media. The spectral and scattering analysis of Klein–Gordon operators yields information about bound states, resonance lifetimes, and energy transport, all of which are relevant in both fundamental physics and applied settings. Moreover, scattering data from Klein–Gordon systems can in principle be used for parameter identification or imaging in layered media, linking the mathematical theory to inverse problems of engineering and medical imaging. Fundamental results of spectral and scattering theories concerning Klein–Gordon operators, both on the whole line and the half line, are discussed in [1, 5, 6, 7, 8, 11, 12, 19].

In recent decades, several authors have extended the spectral and scattering theory to Klein-Gordon operators with impulsive conditions [23]. These studies have revealed that the presence of impulsive conditions can significantly modify the spectral structure of operators. However, the systematic study of the Klein–Gordon operator under the general impulsive conditions, particularly on the half line, remains relatively unexplored. The present paper aims to contribute to this direction by developing the scattering theory for the Klein–Gordon operator on the half line subject to impulsive condition. Using the transfer matrix method, we construct the resolvent operator, establish the correspondence between its poles and the zeros of the components of the transfer matrices and analyze the scattering function through this alternative approach.

This paper is organized as follows: Section 2 presents a brief summary of the fundamental concepts concerning the Klein–Gordon s-wave equation in the absence of impulsive effects, which will serve as a foundational reference for the subsequent analysis. In section 3, we define the transfer matrices associated with the impulsive Klein-Gordon operator and study their analytic dependence on spectral parameter. In section 4, we develop the scattering theory on the half-line, introducing the scattering functions and analyzing their fundamental properties. In the final part of this section, we construct the resolvent operator for the impulsive Klein-Gordon operator and examine its analytic structure. It is shown that the poles of the resolvent coincide with the zeros of a component of the transfer matrices.

2 Preliminaries

Let us consider the Klein-Gordon s-wave equation

$$y'' + [\eta - Q(x)]^2 y = 0, \quad x \in \mathbb{R}_+ := [0, \infty) \quad (1)$$

with the boundary condition

$$y(0) = 0,$$

where Q is a real valued function and η is a spectral parameter. Define $\varphi(x, \eta)$ be a solution of equation (1) that satisfies the initial conditions

$$\varphi(0, \eta) = 0, \quad \varphi'(0, \eta) = 1$$

which can be represented in the form

$$\varphi(x, \eta) = \frac{\sin \eta x}{\eta} + \int_0^x \frac{\sin \eta(x-t)}{\eta} [-Q^2(t) + 2\eta Q(t)] \varphi(t, \eta) dt. \quad (2)$$

It is clear from [12] that $\varphi(x, \eta)$ exists, is unique, and is an entire function with respect to η .

In a similar manner, we construct another solution of equation (1). The result is formulated in the following theorem.

Theorem 1. *The solution $\psi(x, \eta)$ of equation (1) satisfying the conditions*

$$\psi(0, \eta) = 1, \quad \psi'(0, \eta) = 0$$

exists uniquely, is an entire function of η and can also be represented as the solution of the integral equation

$$\psi(x, \eta) = \cos \eta x + \int_0^x \frac{\sin \eta(x-t)}{\eta} [-Q^2(t) + 2\eta Q(t)] \psi(t, \eta) dt. \quad (3)$$

Proof. The solution $\psi(x, \eta)$ admitting the integral form (3) can be derived through the method of variation of parameters. Its existence and uniqueness are subsequently demonstrated via the technique of successive approximations. The concluding statement of the theorem is then justified by applying the Weierstrass convergence criterion for series. $\square$

It is important to observe that

$$W[\varphi, \psi](x, \eta) = -1, \quad \eta \in \mathbb{C},$$

where W refers to the Wronskian of the solutions φ and ψ , which remains constant with respect to x .

Assume that the function Q satisfies the following condition:

$$\int_0^\infty x \{|Q(x)| + |Q'(x)|\} dx < \infty. \quad (4)$$

Provided that condition (4) holds, equation (1) admits the following solutions

$$e^+(x, \eta) = e^{i\alpha(x)} e^{i\eta x} + \int_x^\infty P^+(x, t) e^{i\eta t} dt \quad (5)$$

for $\eta \in \overline{\mathbb{C}}_+$ and

$$e^-(x, \eta) = e^{-i\alpha(x)} e^{-i\eta x} + \int_x^\infty P^-(x, t) e^{-i\eta t} dt \quad (6)$$

for $\eta \in \overline{\mathbb{C}}_-$, where

$$\alpha(x) = \int_x^\infty Q(t) dt, \quad (7)$$

$P^\pm$ are solutions of Volterra type integral equations. In addition, $P^\pm$ and $\frac{\partial}{\partial x}(P^\pm) := P_x^\pm$ satisfy the

inequalities

$$|P^\pm(x, t)| \leq cw \left(\frac{x+t}{2} \right) \exp \{ \gamma(x) \}, \quad (8)$$

$$|P_x^\pm(x, t)| \leq c \left[w^2 \left(\frac{x+t}{2} \right) + \beta \left(\frac{x+t}{2} \right) \right] \quad (9)$$

with a positive constant number c , in which

$$w(x) = \int_x^\infty \{ |Q(t)|^2 + |Q'(t)| \} dt,$$

$$\gamma(x) = \int_x^\infty \{ t |Q(t)|^2 + 2 |Q(t)| \} dt,$$

$$\beta(x) = \frac{1}{4} \{ 2 |Q(x)|^2 + |Q'(x)| \}.$$

By the help of inequality (8), it follows that the solutions e^+ and e^- are analytic with respect to variable η , in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and also remain continuity up to the real axis. Furthermore, both $e^\pm(x, \eta)$ and their derivatives $e_x^\pm(x, \eta)$ satisfy the following asymptotics [8]:

$$\begin{aligned} e^\pm(x, \eta) &= e^{\pm i \eta x} [1 + o(1)], \quad \eta \in \overline{\mathbb{C}}_\pm, \quad x \rightarrow \infty, \\ e_x^\pm(x, \eta) &= e^{\pm i \eta x} [\pm i \eta + o(1)], \quad \eta \in \overline{\mathbb{C}}_\pm, \quad x \rightarrow \infty, \\ e^\pm(x, \eta) &= e^{\pm(i\alpha(x) + i\eta x)} + o(1), \quad \eta \in \overline{\mathbb{C}}_\pm, \quad |\eta| \rightarrow \infty. \end{aligned} \quad (10)$$

It should be emphasized that the following relation

$$W[e^+, e^-](x, \eta) = -2i\eta,$$

is valid for $\eta \in \mathbb{R}$.

3 Construction of the Transfer Matrices

In this section, we will focus on the Klein-Gordon s-wave equation

$$y'' + [\eta - Q(x)]^2 y = 0, \quad x \in \mathbb{R}_+ \setminus \{s\} \quad (11)$$

with the condition

$$y(0) = 0 \quad (12)$$

and under the impulsive condition

$$\begin{bmatrix} y_+(s) \\ y'_+(s) \end{bmatrix} = T \begin{bmatrix} y_-(s) \\ y'_-(s) \end{bmatrix}, \quad T = \begin{bmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{bmatrix}, \quad (13)$$

where

$$y_\pm(s) = \lim_{x \rightarrow s^\pm} y_\pm(x),$$

Q is a real valued function, s is a positive real constant, $\theta_1, \theta_2, \theta_3, \theta_4$ are real numbers such that $\det T \neq 0$ and η is a spectral parameter.

Under the condition (4), equation (11) has the solutions

$$\overline{e^+(x, \eta)} := e^{-i\alpha(x)} e^{-i\eta x} + \int_x^\infty P^-(x, t) e^{-i\eta t} dt \quad (14)$$

for $\eta \in \mathbb{R}$ and

$$\overline{e^-(x, \eta)} := e^{i\alpha(x)} e^{i\eta x} + \int_x^\infty P^+(x, t) e^{i\eta t} dt \quad (15)$$

for $\eta \in \mathbb{R}$, where the function α is defined as in (7), and the functions $P^\pm$ satisfy the inequalities (8) and (9). It should be also emphasized that the following relations

$$\begin{aligned} W[e^+, \overline{e^+}](x, \eta) &= -2i\eta, \\ W[e^-, \overline{e^-}](x, \eta) &= 2i\eta \end{aligned}$$

hold for $\eta \in \mathbb{R}$.

To begin with, we introduce the impulsive operator $\mathcal{L}$ in the Hilbert space

$$L^2[0, \infty) := \left\{ f : \int_0^\infty |f(x)|^2 dx < \infty \right\}$$

by the impulsive problem (11)-(13). We then introduce the transfer matrices associated with the impulsive Klein-Gordon operator $\mathcal{L}$ on the half line and employ them to analyze the scattering functions and their analytic structures through an operator-theoretic way. This approach enables a new characterization of the scattering behavior and establishes connections between the analytic properties of the transfer matrices and the spectral features of the impulsive operator.

Let us denote the solutions of equation (11) on the subintervals $[0, s)$ and (s, ∞) denoting them by y_- and y_+ , respectively:

$$\begin{cases} y_-(x) := y(x), & 0 \leq x < s \\ y_+(x) := y(x), & x > s. \end{cases}$$

With the help of linearly independent solutions of (11) defined on the intervals $[0, s)$ and (s, ∞) , the general form of the solution of (11) for $\eta \in \mathbb{R}$ can be written as

$$\begin{cases} y_-(x, \eta) = A_- \psi(x, \eta) + B_- \varphi(x, \eta), & 0 \leq x < s \\ y_+(x, \eta) = A_+ e^+(x, \eta) + B_+ \overline{e^+(x, \eta)}, & x > s, \end{cases}$$

where the constants $A_\pm$ and $B_\pm$ depend on η . By substituting expressions (2), (3), (5) and (14), we determine the values of $y_-(s, \eta)$, $y_+(s, \eta)$, $\frac{\partial y_-}{\partial x} \Big|_{x=s}$ and $\frac{\partial y_+}{\partial x} \Big|_{x=s}$. Applying the impulsive condition (13), we then obtain the transfer matrix M , which satisfies the following relation:

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = M \begin{bmatrix} A_- \\ B_- \end{bmatrix}, \quad (16)$$

where

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = N^{-1}TD$$

such that

$$N := \begin{bmatrix} e^+(s, \eta) & \overline{e^+(s, \eta)} \\ e_x^+(s, \eta) & \overline{e_x^+(s, \eta)} \end{bmatrix}$$

and

$$D := \begin{bmatrix} \psi(s, \eta) & \varphi(s, \eta) \\ \psi_x(s, \eta) & \varphi_x(s, \eta) \end{bmatrix}.$$

By the help of (16), we have

$$\begin{aligned} M_{11}(\eta) &= -\frac{1}{2i\eta} \left\{ \overline{e_x^+(s, \eta)} [\theta_1 \psi(s, \eta) + \theta_2 \psi_x(s, \eta)] - \overline{e^+(s, \eta)} [\theta_3 \psi(s, \eta) + \theta_4 \psi_x(s, \eta)] \right\}, \\ M_{12}(\eta) &= -\frac{1}{2i\eta} \left\{ \overline{e_x^+(s, \eta)} [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - \overline{e^+(s, \eta)} [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}, \\ M_{21}(\eta) &= -\frac{1}{2i\eta} \left\{ -e_x^+(s, \eta) [\theta_1 \psi(s, \eta) + \theta_2 \psi_x(s, \eta)] + e^+(s, \eta) [\theta_3 \psi(s, \eta) + \theta_4 \psi_x(s, \eta)] \right\}, \\ M_{22}(\eta) &= -\frac{1}{2i\eta} \left\{ -e_x^+(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] + e^+(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}. \end{aligned} \quad (17)$$

Using the components of matrix M , we directly obtain

$$\overline{M_{22}(\eta)} = M_{12}(\eta) \quad (18)$$

for $\eta \in \mathbb{R}$.

Let us consider two distinct solutions of equation (11), denoted by $E^+(x, \eta)$ and $F^+(x, \eta)$, defined as follows:

$$E^+(x, \eta) = \begin{cases} A_-^+ \psi(x, \eta) + B_-^+ \varphi(x, \eta), & 0 \leq x < s \\ A_+^+ e^+(x, \eta) + B_+^+ \overline{e^+(x, \eta)}, & s < x < \infty \end{cases} \quad (19)$$

and

$$F^+(x, \eta) = \begin{cases} A_-^- \psi(x, \eta) + B_-^- \varphi(x, \eta), & 0 \leq x < s \\ A_+^- e^+(x, \eta) + B_+^- \overline{e^+(x, \eta)}, & s < x < \infty, \end{cases} \quad (20)$$

where $A_\pm^\pm$ and $B_\pm^\pm$ are complex coefficients. If $E^+(x, \eta)$ is considered as the solution that can be analytically continued into the upper half plane and $F^+(x, \eta)$ as the solution satisfying the boundary condition (12), then the coefficients satisfy

$$B_+^+ = 0, \quad A_+^+ = 1, \quad A_-^- = 0, \quad B_-^- = k_1, \quad (21)$$

where k_1 is a nonzero real constant. Moreover, by applying the impulsive condition (13) together with (16), the following relations hold uniquely for the same pair of solutions:

$$A_-^+ = \frac{M_{22}(\eta)}{\det M}, \quad B_-^+ = -\frac{M_{21}(\eta)}{\det M}, \quad A_+^- = k_1 M_{12}(\eta), \quad B_+^- = k_1 M_{22}(\eta). \quad (22)$$

Substituting (21) and (22) into (19) and (20), we obtain the following representations for $\eta \in \overline{\mathbb{C}}_+$

$$E^+(x, \eta) = \begin{cases} \frac{M_{22}(\eta)}{\det M} \psi(x, \eta) - \frac{M_{21}(\eta)}{\det M} \varphi(x, \eta), & 0 \leq x < s \\ e^+(x, \eta), & s < x < \infty \end{cases} \quad (23)$$

and for $\eta \in \mathbb{R}$

$$F^+(x, \eta) = \begin{cases} k_1 \varphi(x, \eta), & 0 \leq x < s \\ k_1 M_{12}(\eta) e^+(x, \eta) + k_1 M_{22}(\eta) \overline{e^+(x, \eta)}, & s < x < \infty. \end{cases} \quad (24)$$

Here, It should be particularly emphasized that the functions M_{22} and M_{21} have analytic continuations into the upper half plane. Besides $E^+(x, \eta)$ denotes the Jost solution of the impulsive boundary value problem (11)-(13) while the function

$$E^+(0, \eta) := \frac{M_{22}(\eta)}{\det M}$$

represents the Jost function corresponding to the same problem.

Based on (23) and (24), we can now state the following lemma.

Lemma 1. *For all $\eta \in \mathbb{R}$, the Wronskian of the solutions $E^+(x, \eta)$ and $F^+(x, \eta)$ satisfies*

$$W[E^+, F^+](x, \eta) = \begin{cases} \frac{k_1 M_{22}(\eta)}{\det M}, & x \in [0, s) \\ -2i\eta k_1 M_{22}(\eta), & x \in (s, \infty). \end{cases}$$

Furthermore, let us represent the solutions of equation (11) on the intervals $[0, s)$ and (s, ∞) by $\tilde{y}_-$ and $\tilde{y}_+$, respectively, as follows:

$$\begin{cases} \tilde{y}_-(x) := \tilde{y}(x), & 0 \leq x < s \\ \tilde{y}_+(x) := \tilde{y}(x), & x > s. \end{cases}$$

With the help of linearly independent solutions of (11) defined on the intervals $[0, s)$ and (s, ∞) , the general form of the solution of (11) for $\eta \in \mathbb{R}$ can be written as

$$\begin{cases} \tilde{y}_-(x, \eta) = \tilde{A}_- \psi(x, \eta) + \tilde{B}_- \varphi(x, \eta), & 0 \leq x < s \\ \tilde{y}_+(x, \eta) = \tilde{A}_+ e^-(x, \eta) + \tilde{B}_+ \overline{e^-(x, \eta)}, & x > s, \end{cases}$$

By substituting expressions (2), (3), (6) and (15), we determine the values of $\tilde{y}_-(s, \eta)$, $\tilde{y}_+(s, \eta)$, $\left. \frac{\partial \tilde{y}_-}{\partial x} \right|_{x=s}$ and $\left. \frac{\partial \tilde{y}_+}{\partial x} \right|_{x=s}$. Applying the impulsive condition (13), we then obtain the transfer matrix $\tilde{M}$, which satisfies the following relation:

$$\begin{bmatrix} \tilde{A}_+ \\ \tilde{B}_+ \end{bmatrix} = \tilde{M} \begin{bmatrix} \tilde{A}_- \\ \tilde{B}_- \end{bmatrix}, \quad (25)$$

where

$$\tilde{M} := \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix} = \tilde{N}^{-1} T D$$

such that

$$\tilde{N} := \begin{bmatrix} e^-(s, \eta) & \overline{e^-(s, \eta)} \\ e_x^-(s, \eta) & \overline{e_x^-(s, \eta)} \end{bmatrix}.$$

By the help of (25), we have

$$\begin{aligned}\tilde{M}_{11}(\eta) &= \frac{1}{2i\eta} \left\{ \overline{e_x^-(s, \eta)} [\theta_1 \psi(s, \eta) + \theta_2 \psi_x(s, \eta)] - \overline{e^-(s, \eta)} [\theta_3 \psi(s, \eta) + \theta_4 \psi_x(s, \eta)] \right\}, \\ \tilde{M}_{12}(\eta) &= \frac{1}{2i\eta} \left\{ \overline{e_x^-(s, \eta)} [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - \overline{e^-(s, \eta)} [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}, \\ \tilde{M}_{21}(\eta) &= \frac{1}{2i\eta} \left\{ -e_x^-(s, \eta) [\theta_1 \psi(s, \eta) + \theta_2 \psi_x(s, \eta)] + e^-(s, \eta) [\theta_3 \psi(s, \eta) + \theta_4 \psi_x(s, \eta)] \right\}, \\ \tilde{M}_{22}(\eta) &= \frac{1}{2i\eta} \left\{ -e_x^-(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] + e^-(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}. \quad (26)\end{aligned}$$

Using the components of matrix $\tilde{M}$, we have $\overline{\tilde{M}_{22}(\eta)} = \tilde{M}_{12}(\eta)$ for $\eta \in \mathbb{R}$.

Let us consider two distinct solutions of equation (11), denoted by $E^-(x, \eta)$ and $F^-(x, \eta)$, defined as follows:

$$E^-(x, \eta) = \begin{cases} \tilde{A}_+^+ \psi(x, \eta) + \tilde{B}_+^+ \varphi(x, \eta), & 0 \leq x < s \\ \tilde{A}_+^+ e^-(x, \eta) + \tilde{B}_+^+ \overline{e^-(x, \eta)}, & s < x < \infty \end{cases} \quad (27)$$

and

$$F^-(x, \eta) = \begin{cases} \tilde{A}_-^- \psi(x, \eta) + \tilde{B}_-^- \varphi(x, \eta), & 0 \leq x < s \\ \tilde{A}_-^- e^-(x, \eta) + \tilde{B}_-^- \overline{e^-(x, \eta)}, & s < x < \infty, \end{cases} \quad (28)$$

where $\tilde{A}_\pm^\pm$ and $\tilde{B}_\pm^\pm$ are complex coefficients. If $E^-(x, \eta)$ is considered as the solution that can be analytically continued into the lower half plane and $F^-(x, \eta)$ as the solution satisfying the boundary condition (12), then the coefficients satisfy

$$\tilde{B}_+^+ = 0, \quad \tilde{A}_+^+ = 1, \quad \tilde{A}_-^- = 0, \quad \tilde{B}_-^- = k_2, \quad (29)$$

where k_2 is a nonzero real constant. Moreover, by applying the impulsive condition (13) together with (25), the following relations hold uniquely for the same pair of solutions:

$$\tilde{A}_-^+ = \frac{\tilde{M}_{22}(\eta)}{\det \tilde{M}}, \quad \tilde{B}_-^+ = -\frac{\tilde{M}_{21}(\eta)}{\det \tilde{M}}, \quad \tilde{A}_+^- = k_2 \tilde{M}_{12}(\eta), \quad \tilde{B}_+^- = k_2 \tilde{M}_{22}(\eta). \quad (30)$$

Substituting (29) and (30) into (27) and (28), we obtain the following representations for $\eta \in \overline{\mathbb{C}}_-$

$$E^-(x, \eta) = \begin{cases} \frac{\tilde{M}_{22}(\eta)}{\det \tilde{M}} \psi(x, \eta) - \frac{\tilde{M}_{21}(\eta)}{\det \tilde{M}} \varphi(x, \eta) & 0 \leq x < s \\ e^-(x, \eta), & s < x < \infty \end{cases} \quad (31)$$

and for $\eta \in \mathbb{R}$

$$F^-(x, \eta) = \begin{cases} k_2 \varphi(x, \eta), & 0 \leq x < s \\ k_2 \tilde{M}_{12}(\eta) e^-(x, \eta) + k_2 \tilde{M}_{22}(\eta) \overline{e^-(x, \eta)}, & s < x < \infty. \end{cases} \quad (32)$$

It is virtue to note here that the functions $\tilde{M}_{22}$ and $\tilde{M}_{21}$ have analytic continuations into the lower half plane. Moreover $E^-(x, \eta)$ denotes the Jost solution of the impulsive boundary value problem (11)-(13) while the

function

$$E^-(0, \eta) := \frac{\tilde{M}_{22}(\eta)}{\det \tilde{M}}$$

represents the Jost function corresponding to the same problem.

Based on (31) and (32), we can now state the following lemma.

Lemma 2. For all $\eta \in \mathbb{R}$, the Wronskian of the solutions $E^-(x, \eta)$ and $F^-(x, \eta)$ satisfies

$$W[E^-, F^-](x, \eta) = \begin{cases} \frac{k_2 \tilde{M}_{22}(\eta)}{\det \tilde{M}}, & x \in [0, s) \\ 2i\eta k_2 \tilde{M}_{22}(\eta), & x \in (s, \infty). \end{cases}$$

4 Scattering Properties of Impulsive Operator

In this section, we develop the scattering theory associated with the impulsive Klein–Gordon operator $\mathcal{L}$ defined on the half line. Our main goal is to characterize the scattering functions and the corresponding spectral data in terms of the transfer matrices components introduced in previous section. Then, we construct the resolvent operator corresponding to the impulsive Klein–Gordon operator $\mathcal{L}$ on the half-line and examine its analytic structure. It is shown that the poles of the resolvent coincide with the zeros of a component of the transfer matrices, establishing a precise correspondence between spectral and scattering characteristics.

Now, we can state the theorem required to define the scattering functions of the impulsive Klein–Gordon operator.

Theorem 2. Suppose $\eta \in \mathbb{R} \setminus \{0\}$. Then $M_{22}(\eta) \neq 0$ and $\tilde{M}_{22}(\eta) \neq 0$.

Proof. Suppose there exists a $\eta_0 \in \mathbb{R} \setminus \{0\}$ such that $M_{22}(\eta_0) = 0$. This assumption yields $E^+(0, \eta_0) = 0$. From equation (18), it follows that $\overline{M_{22}(\eta_0)} = M_{12}(\eta_0)$. Consequently, the solution $F^+(x, \eta_0)$ becomes identically zero. However, under this assumption, $F^+(x, \eta_0)$ turns into a trivial solution of (11)-(13) which leads to a contradiction. Therefore, as a result of this contradiction, we conclude that $M_{22}(\eta) \neq 0$ for all $\eta \in \mathbb{R} \setminus \{0\}$.

By proceeding in a similar manner, it is obtained that $\tilde{M}_{22}(\eta) \neq 0$ for all $\eta \in \mathbb{R} \setminus \{0\}$. □

The functions

$$S_1(\eta) := \frac{\overline{E^+(0, \eta)}}{E^+(0, \eta)}$$

and

$$S_2(\eta) := \frac{\overline{E^-(0, \eta)}}{E^-(0, \eta)}$$

are called the scattering functions of the impulsive operator $\mathcal{L}$ for $\eta \in \mathbb{R} \setminus \{0\}$.

By the help of the definitions of scattering functions, we have

$$S_1(\eta) = -\frac{\overline{M_{22}(\eta)}}{M_{22}(\eta)} = \frac{\overline{e_x^+(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^+(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)]}}{e_x^+(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^+(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)]} \quad (33)$$

and

$$S_2(\eta) = -\frac{\overline{\tilde{M}_{22}(\eta)}}{\tilde{M}_{22}(\eta)} = \frac{\overline{e_x^-(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^-(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)]}}{e_x^-(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^-(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)]}. \quad (34)$$

By successively using the equalities (33) and (34), we directly obtain

$$S_1(0) = \lim_{\eta \rightarrow 0} S_1(\eta) = 1$$

and

$$S_2(0) = \lim_{\eta \rightarrow 0} S_2(\eta) = 1,$$

respectively.

Theorem 3. *The scattering functions S_1 and S_2 satisfy the following relations for all $\eta \in \mathbb{R} \setminus \{0\}$,*

$$\overline{S_1(\eta)} = S_1^{-1}(\eta) = S_2(\eta)$$

and

$$\overline{S_2(\eta)} = S_2^{-1}(\eta) = S_1(\eta).$$

Proof. From the definition of $S_1(\eta)$, we obtain

$$\overline{S_1(\eta)} = -\frac{M_{22}(\eta)}{\overline{M_{22}(\eta)}} = S_1^{-1}(\eta)$$

for $\eta \in \mathbb{R} \setminus \{0\}$. Furthermore, after direct calculations, for $\eta \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} \overline{S_1(\eta)} &= \frac{e_x^+(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^+(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)]}{e_x^+(s, \lambda) [\theta_1 \varphi(s, \lambda) + \theta_2 \varphi_x(s, \lambda)] - \overline{e^+(s, \lambda)} [\theta_3 \varphi(s, \lambda) + \theta_4 \varphi_x(s, \lambda)]} \\ &= -\frac{\overline{\tilde{M}_{22}(\eta)}}{\tilde{M}_{22}(\eta)} \\ &= S_2(\eta). \end{aligned}$$

From the last equality, it is obtained directly that

$$\overline{S_2(\eta)} = S_1(\eta)$$

for $\eta \in \mathbb{R} \setminus \{0\}$. On the other side, by proceeding in a similar manner, it is obtained that

$$\overline{S_2(\eta)} = S_2^{-1}(\eta)$$

for $\eta \in \mathbb{R} \setminus \{0\}$. □

Here, we would like to emphasize that mathematically, we have defined two different (they are distinct as long as they take non-real values) functions S_1 and S_2 which are complex conjugates of each other and are based on the upper and lower Jost functions. Physically, these two functions describe the scattering properties of a single system (or potential): S_1 represents the forward scattering process while S_2 corresponds to its time-reversed counterpart. Knowing one of them therefore implies knowing the other and hence the complete scattering information of the system.

Now, we can express the solutions $\hat{e}^+(x, \eta)$ and $\hat{e}^-(x, \eta)$ that will form the basis for determining the resolvent operator of the impulsive Klein–Gordon operator.

Equation (1) admits an unbounded solution $\hat{e}^+(x, \eta)$ satisfying the asymptotic condition

$$\lim_{x \rightarrow \infty} \hat{e}^+(x, \eta) e^{i\eta x} = 1$$

for $\eta \in \overline{\mathbb{C}}_+$. Likewise, there exists another unbounded solution $\hat{e}^-(x, \eta)$ that fulfills

$$\lim_{x \rightarrow \infty} \hat{e}^-(x, \eta) e^{-i\eta x} = 1,$$

for $\eta \in \overline{\mathbb{C}}_-$. It should be emphasized that the following relations hold:

$$\begin{aligned} W[e^+, \hat{e}^+](x, \eta) &= -2i\eta, \\ W[e^-, \hat{e}^-](x, \eta) &= 2i\eta \end{aligned}$$

hold for $\eta \in \overline{\mathbb{C}}_+$ and $\eta \in \overline{\mathbb{C}}_-$, respectively.

Thus, we can consider two distinct unbounded solutions of the impulsive boundary value problem (11)-(13), denoted by $G^+(x, \eta)$ and $G^-(x, \eta)$, defined as follows:

$$G^+(x, \eta) = \begin{cases} \varphi(x, \eta), & 0 \leq x < s \\ a(\eta)e^+(x, \eta) + b(\eta)\hat{e}^+(x, \eta), & s < x < \infty \end{cases} \quad (35)$$

for $\eta \in \overline{\mathbb{C}}_+$ and

$$G^-(x, \eta) = \begin{cases} \varphi(x, \eta), & 0 \leq x < s \\ c(\eta)e^-(x, \eta) + d(\eta)\hat{e}^-(x, \eta), & s < x < \infty \end{cases} \quad (36)$$

for $\eta \in \overline{\mathbb{C}}_-$. With the help of the impulsive condition (13) the following coefficients are determined as:

$$\begin{aligned} a(\eta) &= -\frac{1}{2i\eta} \left\{ \hat{e}_x^+(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - \hat{e}^+(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}, \\ b(\eta) &= \frac{1}{2i\eta} \left\{ e_x^+(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^+(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}, \\ c(\eta) &= \frac{1}{2i\eta} \left\{ \hat{e}_x^-(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - \hat{e}^-(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}, \\ d(\eta) &= -\frac{1}{2i\eta} \left\{ e_x^-(s, \eta) [\theta_1 \varphi(s, \eta) + \theta_2 \varphi_x(s, \eta)] - e^-(s, \eta) [\theta_3 \varphi(s, \eta) + \theta_4 \varphi_x(s, \eta)] \right\}. \end{aligned}$$

By using the solutions (23) and (35) along with the equality $b(\eta) = M_{22}(\eta)$, the following lemma can be expressed.

Lemma 3. For all $\eta \in \overline{\mathbb{C}}_+ \setminus \{0\}$, the Wronskian of the solutions $E^+(x, \eta)$ and $G^+(x, \eta)$ satisfies

$$W[E^+, G^+](x, \eta) = \begin{cases} \frac{M_{22}(\eta)}{\det M}, & x \in [0, s) \\ -2i\eta M_{22}(\eta), & x \in (s, \infty). \end{cases}$$

By using the solutions (31) and (36) along with the equality $d(\eta) = \tilde{M}_{22}(\eta)$, the following lemma can be stated.

Lemma 4. For all $\eta \in \overline{\mathbb{C}}_- \setminus \{0\}$, the Wronskian of the solutions $E^-(x, \eta)$ and $G^-(x, \eta)$ satisfies

$$W[E^-, G^-](x, \eta) = \begin{cases} \frac{\tilde{M}_{22}(\eta)}{\det M}, & x \in [0, s) \\ 2i\eta \tilde{M}_{22}(\eta), & x \in (s, \infty). \end{cases}$$

Let us define the resolvent sets of impulsive Klein-Gordon operator $\mathcal{L}$ as

$$\rho_1(\eta) = \{\eta : \eta \in \mathbb{C}_+, W[E^+, G^+](\eta) \neq 0\}, \quad \rho_2(\eta) = \{\eta : \eta \in \mathbb{C}_-, W[E^-, G^-](\eta) \neq 0\}.$$

Using standard techniques [18], it follows that

$$\rho(\mathcal{L}(\eta)) = \rho_1(\eta) \cup \rho_2(\eta).$$

Consequently, the resolvent operator of $\mathcal{L}$ can be represented as described in the next theorem.

Theorem 4. The resolvent operator of $\mathcal{L}$ is defined as

$$R_\eta(\mathcal{L})\Phi(x) = \int_0^\infty \mathcal{G}(x, t; \eta)\Phi(t)dt$$

for $\eta \in \rho(\mathcal{L}(\eta))$ and $\Phi \in L^2[0, \infty)$, where the Green's function for $\mathcal{L}$ is given by

$$\mathcal{G}(x, t; \eta) = \begin{cases} \mathcal{G}^+(x, t; \eta), & \eta \in \rho_1(\eta) \\ \mathcal{G}^-(x, t; \eta), & \eta \in \rho_2(\eta), \end{cases}$$

where

$$\mathcal{G}^+(x, t; \eta) = \begin{cases} \frac{G^+(t, \eta)E^+(x, \eta)}{W[G^+, E^+](x, \eta)}, & 0 \leq t \leq x, \quad t, x \neq s \\ \frac{E^+(t, \eta)G^+(x, \eta)}{W[G^+, E^+](x, \eta)}, & x \leq t < \infty, \quad t, x \neq s \end{cases}$$

and

$$\mathcal{G}^-(x, t; \eta) = \begin{cases} \frac{G^-(t, \eta)E^-(x, \eta)}{W[G^-, E^-](x, \eta)}, & 0 \leq t \leq x, \quad t, x \neq s \\ \frac{E^-(t, \eta)G^-(x, \eta)}{W[G^-, E^-](x, \eta)}, & x \leq t < \infty, \quad t, x \neq s. \end{cases}$$

By employing classical methods [18], we can define the set of eigenvalues for the impulsive operator $\mathcal{L}$ as

$$\sigma_d(\mathcal{L}) = \{\eta : \eta \in \mathbb{C}_+ \text{ and } W[G^+, E^+](x, \eta) = 0\} \cup \{\eta : \eta \in \mathbb{C}_- \text{ and } W[G^-, E^-](x, \eta) = 0\}.$$

Using Lemma 3, Lemma 4. and components of transfer matrices (17), (26) we can rewrite the set of eigenvalues for the impulsive operator $\mathcal{L}$ as

$$\sigma_d(\mathcal{L}) = \{\eta : \eta \in \mathbb{C}_+ \text{ and } M_{22}(\eta) = 0\} \cup \{\eta : \eta \in \mathbb{C}_- \text{ and } \tilde{M}_{22}(\eta) = 0\}.$$

To analyze the quantitative behavior of the eigenvalues of the impulsive operator $\mathcal{L}$, it is necessary to study the distribution and properties of the zeros of M_{22} and $\tilde{M}_{22}$ in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. For this reason, we can give the following theorem.

Theorem 5. *The asymptotic behavior of the function M_{22} , under condition (4), can be characterized as follows:*

(i) If $\theta_2 \neq 0$, then

$$2i\eta M_{22}(\eta) = \frac{\theta_2}{2} e^{i\alpha(s)} [i\eta + O(1)], \quad \eta \in \overline{\mathbb{C}}_+, \quad |\eta| \rightarrow \infty.$$

(ii) If $\theta_2 = 0$ and $\theta_1 \neq \theta_4$, then

$$2i\eta M_{22}(\eta) = \frac{1}{2} (\theta_1 - \theta_4) e^{i\alpha(s)} [e^{2i\eta s} + O(1)], \quad \eta \in \overline{\mathbb{C}}_+, \quad |\eta| \rightarrow \infty.$$

Proof. The function M_{22} introduced in equation (17) can be expressed in the following form:

$$2i\eta M_{22}(\eta) = \left\{ \theta_1 [e_x^+(s, \eta) e^{-i\eta s}] [\varphi(s, \eta) e^{i\eta s}] + \theta_2 [e_x^+(s, \eta) e^{-i\eta s}] [\varphi_x(s, \eta) e^{i\eta s}] \right. \\ \left. - \theta_3 [e^+(s, \eta) e^{-i\eta s}] [\varphi(s, \eta) e^{i\eta s}] - \theta_4 [e^+(s, \eta) e^{-i\eta s}] [\varphi_x(s, \eta) e^{i\eta s}] \right\}.$$

By employing the asymptotic equality

$$e_x^+(x, \eta) = e^{i\alpha(x)} e^{i\eta x} [i\eta + O(1)], \quad x \in [0, \infty), \quad \eta \in \overline{\mathbb{C}}_+, \quad |\eta| \rightarrow \infty$$

together with relation (10), the last equation takes the form

$$2i\eta M_{22}(\eta) = e^{i\alpha(s)} \left\{ \theta_1 [i\eta + O(1)] \left[\frac{e^{2i\eta s}}{2i\eta} - \frac{1}{2i\eta} + e^{i\eta s} \int_0^s \frac{\sin \eta(s-t)}{\eta} [-Q^2(t) + 2\eta Q(t)] \varphi(t, \eta) dt \right] \right. \\ + \theta_2 [i\eta + O(1)] \left[\frac{e^{2i\eta s}}{2} + \frac{1}{2} + e^{i\eta s} \int_0^s \cos \eta(s-t) [-Q^2(t) + 2\eta Q(t)] \varphi(t, \eta) dt \right] \\ - \theta_3 [1 + o(1)] \left[\frac{e^{2i\eta s}}{2i\eta} - \frac{1}{2i\eta} + e^{i\eta s} \int_0^s \frac{\sin \eta(s-t)}{\eta} [-Q^2(t) + 2\eta Q(t)] \varphi(t, \eta) dt \right] \\ \left. + \theta_4 [1 + o(1)] \left[\frac{e^{2i\eta s}}{2} + \frac{1}{2} + e^{i\eta s} \int_0^s \cos \eta(s-t) [-Q^2(t) + 2\eta Q(t)] \varphi(t, \eta) dt \right] \right\}$$

for $\eta \in \overline{\mathbb{C}}_+$ and $|\eta| \rightarrow \infty$. Hence, if $\theta_2 \neq 0$, the asymptotic form is obtained as

$$2i\eta M_{22}(\eta) = \frac{\theta_2}{2} e^{i\alpha(s)} [i\eta + O(1)]$$

for $\eta \in \overline{\mathbb{C}}_+$ and $|\eta| \rightarrow \infty$. If $\theta_2 = 0$ and $\theta_1 \neq \theta_4$, then it is obtained as

$$2i\eta M_{22}(\eta) = \frac{1}{2} (\theta_1 - \theta_4) e^{i\alpha(s)} [e^{2i\eta s} + O(1)]$$

for $\eta \in \overline{\mathbb{C}}_+$ and $|\eta| \rightarrow \infty$. This completes the proof. $\square$

Theorem 6. *The asymptotic behavior of the function $\tilde{M}_{22}$, under condition (4), can be characterized as follows:*

(i) If $\theta_2 \neq 0$, then

$$2i\eta \tilde{M}_{22}(\eta) = -\frac{\theta_2}{2} e^{-i\alpha(s)} [-i\eta + O(1)], \quad \eta \in \overline{\mathbb{C}}_-, \quad |\eta| \rightarrow \infty.$$

(ii) If $\theta_2 = 0$ and $\theta_1 \neq \theta_4$, then

$$2i\eta M_{22}(\eta) = \frac{1}{2} (\theta_4 - \theta_1) e^{-i\alpha(s)} [e^{-2i\eta s} + O(1)], \quad \eta \in \overline{\mathbb{C}}_-, \quad |\eta| \rightarrow \infty.$$

Proof. By proceeding in a similar manner, the proof is obtained directly. $\square$

As a direct consequence of Theorem 5 and Theorem 6, the following theorem can be stated.

Theorem 7. Assume that $\theta_2 \neq 0$ or $\theta_1 \neq \theta_4$. Under the condition (4), the set of eigenvalues of $\mathcal{L}$ are bounded with at most countably many elements, and their limit points are restricted to a bounded subinterval on the real axis.

Proof. From Theorem 5, it follows that for sufficiently large $\eta \in \mathbb{C}_+$, we have $M_{22}(\eta) \neq 0$. Consequently, the zeros of M_{22} in $\mathbb{C}_+$ is confined to a limited region, which ensures that the set eigenvalues of $\mathcal{L}$ is bounded in $\mathbb{C}_+$. Similarly, Theorem 6 shows that $\tilde{M}_{22}(\eta)$ does not vanish for sufficiently large $\eta \in \mathbb{C}_-$, implying the boundedness of the set eigenvalues of $\mathcal{L}$ in $\mathbb{C}_-$. Furthermore, since $M_{22}(\eta)$ and $\tilde{M}_{22}(\eta)$ are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, the set eigenvalues of $\mathcal{L}$ contains at most countably many points, with their limit points restricted to a bounded subinterval of $\mathbb{R}$. $\square$

5 Conclusion

In this study, we have investigated the scattering characteristics of impulsive Klein–Gordon operators on the half line, under general impulsive condition. By constructing analytic representations of the Jost type solutions and scattering functions, we have derived explicit formulations for the resolvent operator and provided asymptotic relations for the transfer matrices components. The obtained results describe how the impulsive conditions affect the scattering structure, determining the boundedness of the eigenvalues set.

This work not only contributes to the understanding of the scattering behavior of impulsive operators but also offers new insights into the application of transfer matrix technique in scattering theory. The results presented here will serve as a foundation for future studies on the scattering theory of more complex impulsive systems. Further research could explore the broader implications of these results, extending the methodology first to impulsive Klein-Gordon operators on the whole real axis, and then to other types of differential operators.

Author Contribution Statements

All parts of the study were conducted solely by the author.

Declaration of Competing Interests

The author declares that there are no competing financial or personal interests.

Data Availability Statement

This manuscript has no associated data.

Funding Source Declaration

No funding was received for conducting this study.

Ethical Standard

This article does not contain any studies with human participants or animals performed by the author.

Declaration of Generative Artificial Intelligence Use

The author declares that no generative Artificial Intelligence tools were used in the preparation of this manuscript.

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