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The general Racah algebra as the symmetry algebra of generic systems on pseudo–spheres

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Abstract

We characterize the symmetry algebra of the generic superintegrable system on a pseudo-sphere corresponding to the homogeneous space $SO(p,q+1)/SO(p,q)$ where $p+q=\mathcal{N}$, $\mathcal{N}\in\mathbb{N}$. These symmetries occur both in quantum as well as in classical systems in various contexts, so they are quite important in physics. We show that this algebra is independent of the signature $(p,q+1)$ of the metric and that it is the same as the Racah algebra $\mathcal{R}(\mathcal{N}+1)$. The spectrum obtained from $\mathcal{R}(\mathcal{N}+1)$ via the Daskaloyannis method depends on undetermined signs that can be associated to the signatures. Two examples are worked out explicitly for the cases $SO(2,1)/SO(2)$ and $SO(3)/SO(2)$ where it is shown that their spectrum obtained by means of separation of variables coincide with particular choices of the signs, corresponding to the specific signatures, of the spectrum for the symmetry algebra $\mathcal{R}(3)$.

KEYWORDS: Racah algebra, symmetry algebra, pseudo-spheres, superintegrable system, spectrum

1 Introduction

The Racah algebra $\mathcal{R}(3)$ has been applied to describe the recoupling of three copies of $su(1,1)$, but it has been used in many other contexts, for instance it was shown that it is the symmetry algebra of the so called "generic superintegrable system" on the sphere $\mathcal{S}^2$

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[1, 2, 3, 4]. Another related property is that this algebra can be identified as the commutant of $o(2) \oplus o(2) \oplus o(2)$ in $o(6)$. All these properties can be extended to $\mathcal{N}$ copies of $su(1,1)$ and then it is called the generalized $\mathcal{R}(\mathcal{N})$ Racah algebra [5, 6, 2, 7]. The connection with the Howe duality and embedding into Bannai-Ito algebra was also discussed. The Racah algebra $\mathcal{R}(3)$ which is included in Askey-Wilson QAW(3) algebra [8] has been also applied in position dependent mass systems [9]. Therefore, Racah algebras have a high interest in classical and quantum physics as well as in mathematical physics.

Along this work we will consider the symmetry algebra of generic systems defined on a pseudo–sphere in an ambient space $\mathbb{R}^{p+q+1}$ endowed with a pseudo Riemannian metric $g^{\mu\nu}$ with signature $(p, q+1)$ which is assumed to be in the canonical diagonal form. Other notations can be used, but in order to fix a kind of systems to be considered later let us define a pseudo–sphere $\mathcal{S}^{p,q}$ through the equation

$$s_1{}^2 + \cdots + s_p{}^2 - s_{p+1}{}^2 - \cdots - s_{p+q+1}{}^2 = -1, \qquad \mathbf{s} \in \mathbb{R}^{p+q+1}\,. \tag{1.1}$$

This is an orbit of the pseudo orthogonal group $SO(p, q+1)$ in the ambient space $\mathbb{R}^{p+q+1}$, and it can also be seen as the homogeneous space $\mathcal{S}^{p,q} = SO(p, q+1)/SO(p, q)$. We shall see that all the symmetry algebras of the "generic systems" associated to any such metric $g^{\mu\nu}$ with $p + q = \mathcal{N}$ fixed, will coincide. This means that their symmetry algebras can be identified to the same Racah algebra $\mathcal{R}(\mathcal{N}+1)$. This algebra is the commutant of $\oplus^{\mathcal{N}+1} o(2)$ in the Lie algebra $o(2p, 2q+2)$ [10, 11], which should be the isomorphic to the commutant in $so(2\mathcal{N}+2)$, i.e., independent of the signature.

We will analyse the possible discrete spectra for the particular case $\mathcal{R}(3)$, by following the method of Daskaloyannis [12]. Then, we will show that the formulas so obtained include the two cases which have discrete spectrum corresponding to the systems defined on the sphere $\mathcal{S}^2 \equiv \mathcal{S}^{0,2} \approx SO(3)/SO(2)$ and on the (two-sheeted) hyperbolic space $\mathcal{H}^2 \equiv \mathcal{S}^{2,0} \approx SO(2,1)/SO(2)$. The signature of the initial Hamiltonian can be identified in the final formula of the spectrum.

The organization of the paper is as follows. In Section 2, we define this kind of generic superintegrable systems on a general pseudo–sphere. We supply the form of the quadratic symmetries and the symmetry algebra which is independent of the metric coefficients $g^{\mu\nu}$. It is also included the quantum coefficient $\hbar$ in all the terms so that the classical limit of the system is obtained by taking $\hbar \to 0$. Next, in Section 3, the symmetry algebra for $\mathcal{N} = 3$ is written in the form of a Daskaloyannis algebra [13]. In this way we have computed the possible discrete spectrum of the symmetry algebra. In Section 4, we supply the spectrum of the quantum systems defined on the sphere $\mathcal{S}^2$ [14] and on the hyperboloid $\mathcal{H}^2$ [10], which can be obtained by means of separation of variables. We will check that, indeed, these formulas are included in the ones obtained in Section 3. The paper will end with some remarks and conclusions in Section 4.

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2 Generalized Racah Algebra

Let the pseudo–sphere $\mathcal{S}^{p,q}$ be defined as the surface

$$g_{ij}s^i s^j = -1, \qquad (s^1, \ldots, s^{\mathcal{N}+1}) \in \mathbb{R}^{\mathcal{N}+1} \tag{2.1}$$

where the g_{ik}'s are metric coefficients $g = (g_{ik})$ with signature $(p, q+1)$ as mentioned above. Then, the Hamiltonian of a $\mathcal{N} = p + q$ dimensional superintegrable system on $\mathcal{S}^{p,q}$ is defined by

$$H = \frac{1}{2}\, g_{ik}g_{jl}J_{ij}J_{kl} + g_{ii}\frac{a_i}{s_i^2}, \qquad a_i \in \mathbb{R} \tag{2.2}$$

where

$$J_{ij} = \hbar g_{jk}s_i\partial_k - \hbar g_{ik}s_j\partial_k \tag{2.3}$$

are the anti-Hermitian generators of the Lie algebra $SO(p, q+1)$ which leaves invariant the pseudo–sphere. We have also included the quantum constant $\hbar$ in order to consider later the classical limit. We use the convention of sum in the repeated indexes with some care: if one of the repeated indexes is in both sides of the equation and it is taking part of the definition of a component of a tensor, it will not be summed.

The symmetries for the generic Euclidean case have been known from some time ago [15]. In the case of generic metric $g^{\mu\nu}$ (see Ref. [11] were the signature was considered) the second order symmetries of the above Hamiltonian (2.2)have the form

$$Q_{ij} = -g_{ik}g_{jl}J_{ij}J_{kl} + g_{ii}g_{jj}a_i\frac{s_j^2}{s_i^2} + g_{jj}g_{ii}a_j\frac{s_i^2}{s_j^2}\,. \tag{2.4}$$

The commutations of these symmetries are as follows. Firstly, the commutators

$$[Q_{ij}, Q_{ik}] = C_{ijk}\,, \tag{2.5}$$

will lead to the third order symmetry operators C_{ijk}. All these operators $\{Q_{ij}, C_{klm}\}$ will close a quadratic symmetry algebra:

$$\begin{aligned}
[Q_{jk},, C_{ijk}] &= 8\hbar^2 Q_{ik}Q_{jk} - 8\hbar^2 Q_{jk}Q_{ij} + 8(-\hbar^4 + 2a_j\hbar^2)Q_{ik} \\
&\quad -8(-\hbar^4 + 2a_k\hbar^2)Q_{ij} + 8(a_j - a_k)\hbar^4\,, \\
[Q_{kl}, C_{ijk}] &= 8\hbar^2 Q_{ik}Q_{jl} - 8\hbar^2 Q_{il}Q_{jk} + 4\hbar^4 Q_{ik} + 4\hbar^4 Q_{jl} - 4\hbar^4 Q_{il} - 4\hbar^4 Q_{jk}\,, \\
[C_{ijk}, C_{jkl}] &= -8\hbar^2 C_{jkl}Q_{ij} - 8\hbar^2 C_{ikl}Q_{jk} - 8\hbar^2 C_{ijk}Q_{jl} \\
&\quad +4\hbar^4 C_{jkl} - 4\hbar^4 C_{ijk} - 8(-\hbar^4 + 2a_j\hbar^2)C_{ikl}\,, \\
[C_{ijk}, C_{klm}] &= -8\hbar^2 C_{ilm}Q_{jk} - 8\hbar^2 Q_{ik}C_{jlm} + 4\hbar^4 C_{ilm} - 4\hbar^4 C_{jlm}\,,
\end{aligned} \tag{2.6}$$

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$$[C_{ijk}, C_{lmn}] = 0\,.$$

Any other not stated commutator is vanishing. These symmetry operators are invariant under cyclic permutation of their subindexes,

$$Q_{ij} = Q_{ji}, \qquad C_{ijk} = C_{kij} = C_{jki}\,.$$

Remark that the Hamiltonian can be expresed in terms of the quadratic symmetries Q_{ij}:

$$\sum_{i<j} Q_{ij} - H + \sum_{i}^{n} a_i = 0 \tag{2.7}$$

Therefore, we see that the signature in the initial Hamiltonian determine the constants of motion (2.4), but the symmetry algebra (2.6) does not include any track of the metric coefficients g^{ij}, hence it is the same for any generic system (2.2) on a pseudo Riemannian surface.

2.1 Classical case

In the previous section the quantum symmetry algebra has terms with the quantum constant $\hbar$. In order to get the symmetry algebra of the corresponding classical system (replacing the quantum by classical magnitudes in the initial Hamiltonian) it is enough to take the limit $\hbar \to 0$ and replace commutators by Poisson brackets (PB). In this way, we get the following algebra:

$$\{Q_{ij}, Q_{ik}\}_{PB} = C_{ijk}\,, \tag{2.8}$$

$$\begin{aligned}
&\{Q_{jk,}, C_{ijk}\}_{PB} = -8Q_{ik}Q_{jk} + 8Q_{jk}Q_{ij} - 16a_jQ_{ik} + 16a_kQ_{ij}\,,\\
&\{Q_{kl}, C_{ijk}\}_{PB} = -8Q_{ik}Q_{jl} + 8Q_{il}Q_{jk}\,,\\
&\{C_{ijk}, C_{jkl}\}_{PB} = 8C_{jkl}Q_{ij} + 8C_{ikl}Q_{jk} + 8C_{ijk}Q_{jl} + 16a_jC_{ikl}\,,\\
&\{C_{ijk}, C_{klm}\}_{PB} = 8C_{ilm}Q_{jk} + 8Q_{ik}C_{jlm}\,,\\
&\{C_{ijk}, C_{lmn}\}_{PB} = 0\,.
\end{aligned} \tag{2.9}$$

Other Poisson brackets not stated are vanishing. The symmetries satisfy the same cyclic relations as in the quantum case:

$$Q_{ij} = Q_{ji}, \qquad C_{ijk} = C_{kij} = C_{jki}\,.$$

If we compare the classical and the quantum symmetry algebras, we can appreciate that, as usual, the classical one is simpler because some of the terms in the quantum commutators vanish in the classical limit.

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3 An Example: The Three–Dimensional Case

The set of of second order symmetries in the three-dimensional case are Q_{12}, Q_{13}, Q_{23} (for more details of the examples on the spheres S^2 and S^3 see Ref. [6]). As the system is superintegrable there must be three independent symmetries including the Hamiltonian. Thus, we can choose two of them Q_{12}, Q_{13} together with H as the independent set. The other one Q_{23}, with the help of (2.7), can be expressed as

$$Q_{23} = -Q_{12} - Q_{13} + H - \sum a_i \,.$$

The only third order symmetry of the type C_{ijk} is C_{213}. In this section we will take $\hbar = 1$ since we will not consider the classical counterpart. Then, the set of the three symmetries $\{Q_{12}, Q_{13}, C_{213}\}$ close, together with (2.5), the following algebra,

$$\begin{aligned}
[Q_{12}, C_{213}] &= +8\{Q_{12}, Q_{13}\} + 16(-1 + a_1 + a_2)Q_{13} \\
&\quad +4(-2 + 6a_1 + 2a_2 + 2a_3)Q_{12} - 8(-1 + 2a_1)H \\
&\quad -8HQ_{12} + 8Q_{12}^2 + 4(-4a_1 + 4a_1^2 + 4a_1a_2 - 2a_3 + 4a_1a_3)\,, \\
[Q_{13}, C_{213}] &= -8\{Q_{12}, Q_{13}\} - 8Q_{13}^2 + 8HQ_{13} \\
&\quad -4(-2 + 6a_1 + 2a_2 + 2a_3)Q_{13} - 16(-1 + a_1 + a_3)Q_{12} \\
&\quad +8(-1 + 2a_1)H - 4(-4a_1 + 4a_1^2 - 2a_2 + 4a_1a_2 + 4a_1a_3)\,.
\end{aligned} \tag{3.1}$$

This algebra can be rewritten in the form of a Daskaloyannis type algebra [12, 13] spanned by the generators $\{A, B, C\}$ and having the commutations in the form

$$\begin{aligned}
[A, B] &= C\,, \\
[A, C] &= \alpha A^2 + \gamma\{A, B\} + \delta A + \epsilon B + \zeta\,, \\
[B, C] &= aA^2 - \gamma B^2 - \alpha\{A, B\} + dA - \delta B + z\,,
\end{aligned} \tag{3.2}$$

where the structure constants in our case take the values:

$$\alpha = 8, \quad \gamma = 8, \quad \epsilon = 16, \quad a = 0, \quad \delta = 4(-2 + 6a_1 + 2a_2 + 2a_3) - 8H,$$
$$d = -16(-1 + a_1 + a_3), \quad \zeta = 4(-4a_1 + 4a_1^2 + 4a_1a_2 - 2a_3 + 4a_1a_3) - 8(-1 + 2a_1)H,$$
$$z = 8(-1 + 2a_1)H - 4(-4a_1 + 4a_1^2 - 2a_2 + 4a_1a_2 + 4a_1a_3)\,.$$

The Casimir of this algebra is given by

$$\begin{aligned}
K &= C^2 - \alpha\{A^2, B\} - \gamma\{A, B^2\} + (\alpha, \gamma - \delta)\{A, B\} \\
&\quad +(\gamma^2 - \epsilon)B^2 + (\gamma\delta - 2\zeta)B + \frac{2a}{3}A^3 + (d + \frac{a\gamma}{3} + \alpha^2)A^2 + (\frac{a\epsilon}{3} + \alpha\delta + 2z)A\,,
\end{aligned} \tag{3.3}$$

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which can be written in the present realization in terms of the Hamiltonian:

$$\begin{aligned} K = {} & 4(-3+4a_1)H^2 - 8(6-21a_1+4a_1^2-3a_2+4a_1a_2-3a_3+4a_1a_3)H \\ & +4\big(20a_1-39a_1^2+4a_1^3+4a_2-30a_1a_2 \\ & +8a_1^2a_2-3a_2^2+4a_1a_2^2+4a_3-30a_1a_3+8a_1^2a_3 \\ & +6a_2a_3-8a_1a_2a_3-3a_3^2+4a_1a_3^2\big)\,. \end{aligned} \tag{3.4}$$

This formula, providing a link between the Casimir and the Hamiltonian, will allow to establish the realization of the quadratic algebra as a deformed oscillator algebra [12] of the form

$$[N,b] = -b, \quad [N,b^\dagger] = b^\dagger, \quad bb^\dagger = \Phi(N+1), \quad b^\dagger b = \Phi(N)$$

where $\Phi(N)$ is the structure function which is a polynomial in terms of the number operator N and the representation dependent parameter u. Remark that this algebraic approach has been extended to polynomial algebras [16] with three generators and applied to higher rank quadratic algebras [17]. The expression of Φ is

$$\begin{aligned} \Phi(N) = {} & 768\left(\alpha\epsilon^2+4\gamma^2\zeta-2\gamma\delta\epsilon\right)^2+32\gamma^4(2(N+u)-1)^2 \\ & \big(12(N+u)^2-12(N+u)-1\big)\left(3\alpha^2\epsilon^2+4\alpha\gamma^2\zeta-6\alpha\gamma\delta\epsilon+2a\gamma\epsilon^2\right. \\ & \left.+2\gamma^2\delta^2-4\gamma^2d\epsilon+8\gamma^3z-48\gamma^6\right)(2(N+u)-3)(2(N+u)-1)^4(2(N+u)+1)\left(\alpha^2\epsilon-\alpha\gamma\delta\right. \\ & \left.+a\gamma\epsilon-\gamma^2d\right)-256\gamma^2(2(N+u)-1)^2\left(3\alpha^2\epsilon^3+4\alpha\gamma^4\zeta\right. \\ & +12\alpha\gamma^2\zeta\epsilon-9\alpha\gamma\delta\epsilon^2+a\gamma\epsilon^3+2\gamma^4\delta^2 \\ & \left.-12\gamma^3\delta\zeta+6\gamma^2\delta^2\epsilon+2\gamma^4d\epsilon-3\gamma^2d\epsilon^2-4\gamma^5z+\right. \\ & \left.12\gamma^3z\epsilon\right)+\gamma^8(2(N+u)-3)^2(2(N+u)-1)^4(2(N+u)+1)^2\left(3\alpha^2+4a\gamma\right)-3072\gamma^6K(2(N+u)-1)^2\,. \end{aligned}$$

Using the structure constants and the Casimir operator as expressed in terms of the central element of the algebra (the Hamiltonian H) one get

$$\begin{aligned} \Phi(N,u,E) = {} & 3221225472(a_1^2+a_2^2-2a_1(a_2+(1-2(N+u))^2) \\ & -2a_2(1-2(N+u))^2+4(1-2(N+u))^2(-1+(N+u))(N+u) \\ & (a_3^2+E^2-2a_3(E+(1-2(N+u))^2) \\ & -2E(1-2N)^2+4(1-2(N+u))^2(-1+(N+u))(N+u))\,. \end{aligned} \tag{3.5}$$

The structure function in this form allow us to characterize the finite dimensional unitary representations and in this way it will lead to the discrete spectrum for the energy

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E. This formula is a polynomial of degree 8 in the number operator N, but it is not yet in a convenient form. Using the parameter

$$m_i^2 = 1 + 4a_i, \quad i = 1, 2, 3$$

and introducing

$$-\tilde{E}^2 = -1 + 4E\,,$$

we can reexpress the structure function as

$$\Phi(N, u, E) = 824633720832(N + u - N_1)(N + u - N_2) \tag{3.6}$$
$$(N + u - N_3)(N + u - N_4)(N + u - N_5)$$
$$(N + u - N_6)(N + u - N_7)(N + u - N_8)$$

with

$$N_1 = \tfrac{1}{4}(2 - (m_1 - m_2)), \qquad N_5 = \tfrac{1}{4}(2 - (\tilde{E} - m_3)),$$
$$N_2 = \tfrac{1}{4}(2 + (m_1 - m_2)), \qquad N_6 = \tfrac{1}{4}(2 + (\tilde{E} - m_3)),$$
$$N_3 = \tfrac{1}{4}(2 - (m_1 + m_2)), \qquad N_7 = \tfrac{1}{4}(2 - (\tilde{E} + m_3)),$$
$$N_4 = \tfrac{1}{4}(2 + (m_1 + m_2)), \qquad N_8 = \tfrac{1}{4}(2 + (\tilde{E} + m_3))\,.$$

This is a factorized form that will facilitate greatly the study of the finite dimensional unitary representations. The constraints that need to be satisfied in order to get finite dimensional unitary representations are the following:

$$\Phi(0, u, E) = 0, \quad \Phi(p + 1, u, E) = 0, \quad \Phi(\nu, u, E) > 0 \qquad \forall\, \nu = 1, ..., p\,.$$

The first condition $\Phi(0, u, E) = 0$ provide

$$u = \frac{1}{4}(2 + m_1\epsilon_1 + m_2\epsilon_2)$$

where the parameters $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ allow to describe the different solutions in a unified way. They supply us with the structure function under the form

$$\Phi(N, E) = 12582912(-4N + m_1(-1 - \epsilon_1) + m_2(-1 - \epsilon_2)(-4N + m_1(1 - \epsilon_1) + m_2(-1 - \epsilon_2)$$
$$(-4N + m_1(-1 - \epsilon_1) + m_2(1 - \epsilon_2)(-4N + m_1(1 - \epsilon_1) + m_2(1 - \epsilon_2)$$
$$(-\tilde{E} - 4N - m_3 + m_1\epsilon_1 - m_2\epsilon_2)(\tilde{E} - 4N - m_3 + m_1\epsilon_1 - m_2\epsilon_2)$$
$$(-\tilde{E} - 4Nm_3 + m_1\epsilon_1 - m_2\epsilon_2)(\tilde{E} - 4Nm_3 - m_1\epsilon_1 - m_2\epsilon_2)\,.$$

The second condition $\Phi(p + 1, u, E) = 0$ provides

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$$\tilde{E} = 4(p+1) - \epsilon_3 m_3 - m_2\epsilon_2 - m_1\epsilon_1$$

and

$$E = \frac{1}{4} - \frac{1}{4}(\epsilon_3 m_3 + \epsilon_2 m_2 + \epsilon_1 m_1 - 4(p+1))^2 .$$

We will introduce the new parameters l_i whose meaning will be explained below:

$$m_i = 2\, l_i, \quad i = 1, 2, 3 .$$

Then, the spectrum will take the form

$$E = \frac{1}{4} - (\epsilon_3 l_3 + \epsilon_2 l_2 + \epsilon_1 l_1 - 2(p+1))^2 \tag{3.7}$$

where $a_i = l_i^2 - 1/4$. The degeneracy of each energy level, determined by p, is $p+1$.

3.1 The spectrum of the three dimensional system

We will consider an example corresponding to metric coefficients of different signs: ($g_{11} = +1, g_{22} = +1, g_{33} = -1$). The surface is the hyperbolic space $\mathcal{H}^2$, the upper component of the two dimensional two-sheeted hyperboloid given by $s_1^2 + s_2^2 - s_3^2 = -1$, inside the real space $\mathbb{R}^3$.

The superintegrable generic Hamiltonian in this case has the form [10]:

$$H = -J_1^2 - J_2^2 + J_3^2 + \frac{\ell_1^2 - \frac{1}{4}}{s_1^2} + \frac{\ell_2^2 - \frac{1}{4}}{s_2^2} - \frac{\ell_3^2 - \frac{1}{4}}{s_3^2} , \tag{3.8}$$

where $\ell = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3$, and the anti-Hermitian generators of $SO(2,1)$ are

$$\begin{aligned} J_1 &= s_2\, \partial_{s_3} + s_3\, \partial_{s_2} , \\ J_2 &= s_3\, \partial_{s_1} + s_1\, \partial_{s_3} , \\ J_3 &= -s_1\, \partial_{s_2} + s_2\, \partial_{s_1} . \end{aligned} \tag{3.9}$$

Here, J_3 generates true rotations around s_3, while J_1, J_2 are generators of the pseudo-rotations around s_1 and s_2, respectively. In terms of these generators the kinetic part of the Hamiltonian (3.8) is (proportional to) the $so(2,1)$ Casimir operator

$$C = J_1^2 + J_2^2 - J_3^2 . \tag{3.10}$$

We can parametrize the hyperbolic surface in the following way [10]:

$$s_1 = \sinh\xi\, \cos\theta , \qquad s_2 = \sinh\xi\, \sin\theta , \qquad s_3 = \cosh\xi , \tag{3.11}$$

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where $0 \leq \theta < 2\pi$, $0 \leq \xi < \infty$. Using this parametrization, the Hamiltonian (3.8), takes the form

$$H = -\partial_\xi^2 - \coth\xi\,\partial_\xi - \frac{\ell_3^2 - \frac{1}{4}}{\cosh\xi^2} + \frac{1}{\sinh\xi^2}\left(-\partial_\theta^2 + \frac{\ell_2^2 - \frac{1}{4}}{\cos\theta^2} + \frac{\ell_1^2 - \frac{1}{4}}{\sin\theta^2}\right). \tag{3.12}$$

The corresponding eigenvalue equation is $H\,\Psi(\xi,\theta) = E\,\Psi(\xi,\theta)$.

It can be shown by separating variables [10] that the discrete spectrum is given by

$$E = \frac{1}{4} - (\ell_3 - \ell_1 - \ell_2 - 2(n+m+1))^2, \tag{3.13}$$

where $m, n \in \mathbb{Z}^+$ and $\ell_3 - \ell_1 - \ell_2 - 2(n+m+1) > 0$. The degeneracy is given by the $P+1$ values of m, n such that $m + n = P$, with $0 \leq P < (\ell_3 - \ell_1 - \ell_2)/2 - 1$.

We can compare this formula with that obtained following the Daskaloyannis method given by (3.7). We see that they will coincide if we make the identifications $g_{ii} = \epsilon_i$ and $p = n + m \equiv P$. The values of $p = 0, 1\ldots$ are subject to the condition

$$\frac{\ell_3 - \ell_1 - \ell_2}{2} - 1 > p \geq 0\,. \tag{3.14}$$

The allowed values of p determine the finite spectrum and the degeneracy of each level is $p + 1$. Only the ground level $p = 0$ will be a singlet.

Let us briefly mention the case with metric coefficients of equal signs: $g_{ii} = 1$, $i = 1, 2, 3$. The surface is $\mathcal{S}^2$, the two dimensional sphere, $s_1^2 + s_2^2 + s_3^2 = 1$ in $\mathbb{R}^3$ (equivalently we could take $g_{ii} = -1$ and the equation of the sphere with an overall -1 sign). The generic Hamiltonian in this case has the usual form [14]:

$$H = -(J_1^2 + J_2^2 + J_3^2) + \frac{\ell_1^2 - \frac{1}{4}}{s_1^2} + \frac{\ell_2^2 - \frac{1}{4}}{s_2^2} + \frac{\ell_3^2 - \frac{1}{4}}{s_3^2}\,, \tag{3.15}$$

where $\ell = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3$, and the anti-Hermitian generators of $SO(3)$. The spectrum is given by

$$-E = \frac{1}{4} - (\ell_1 + \ell_2 + \ell_3 + 2(n+m+1))^2\,, \tag{3.16}$$

where $m, n \in \mathbb{Z}^+$. There is an infinite number of discrete energy levels whose degeneracy also is given by the $P + 1$ values of m, n such that $m + n = P$. The formula (3.7) applies also here provided we take $\epsilon_i = -1$, $i = 1, 2, 3$, as well as an overall change of sign coming from the initial Hamiltonian (3.15).

The generalization to examples of higher dimensions in principle can be done, but it is a difficult task. The main problem is that the method of Daskaloyannis need to be extended to higher rank algebras, for instance along the lines shown in [16, 17]. On the other hand, the spectrum, degeneracy and eigenfunctions can be obtained by separation of variables in different ways [18, 14, 10, 11].

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4 Conclusions

We derived the symmetry algebra of the generic Hamiltonian on the pseudo–sphere $\mathcal{S}^{p,q}$ corresponding to an ambient space $\mathbb{R}^{p+q+1}$ with metric $g^{\mu\nu}$ of signature $(p, q+1)$ (in canonical diagonal form). We showed that this algebra is the same, independently of the metric, as the general Racah algebra $\mathcal{R}(\mathcal{N}+1)$, where $\mathcal{N} = p+q$. For example, our formula (2.7) coincides with (7) of [2], and our commutation rules (2.6) correspond to (9a)-(9e) in [2]. In other words, we are dealing with the same algebra but different (affinely related) basis.

We considered in detail the particular case $g^{\mu\nu} = \mathrm{diag}(1, 1, -1)$, the homogeneous space $SO(2,1)/SO(2)$ of the two sheeted hyperbolic space and the corresponding generic Hamiltonian. We constructed the symmetry algebra, its Casimir operator, the realization as a deformed oscillator algebra and calculated the energy spectrum algebraically which depend on some signs (ϵ_i). We compared this algebraic spectrum with the "physical spectrum" obtained via separation of variables of the corresponding Schrödinger equation and showed how they both coincide for a choice of the signs which is given by the signature of the metric. In Ref. [10] it was shown that the symmetry algebra of each of the generic Hamiltonians in $SO(2,1)/SO(2)$ can be identified as the commutant of $\oplus^3 so(2)$ in the enveloping algebra of $so(4,2)$. Meanwhile, in the standard case of the generic system on $SO(3)/SO(2)$ it is known, based on the Howe duality, that the symmetry algebra is identified as the commutant of $\oplus^3 o(2)$ in the enveloping algebra of $o(6)$. It seems as if the symmetry operators, being quadratic, loose the track of the signature and they have the same algebraic structure for any metric signature. There are even some homogeneous spaces, for instance $SO(2,1)/SO(1,1)$, where the inner product, in the space of wave functions on the manifold, is not positive definite and therefore there is not the notion of bound states. However, in this case, the algebra would be the same as in the above cases because the homogeneous space is not used in the derivation of the algebra.

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