Bulletin of the Iranian Mathematical Society
https://doi.org/10.1007/s41980-021-00605-8

ORIGINAL PAPER

On a Symmetric Generalization of Bivariate Sturm–Liouville Problems

Yves Guemo Tefo[1,2] · Rabia Aktaş[3] · Iván Area[1] · Esra Güldoğan Lekesiz[4]

Received: 13 February 2021 / Revised: 9 June 2021 / Accepted: 14 June 2021
© The Author(s) 2021, corrected publication 2021

Abstract
A new class of partial differential equations having symmetric orthogonal solutions is presented. The general equation is presented and orthogonality is obtained using the Sturm–Liouville approach. Conditions on the polynomial coefficients to have admissible partial differential equations are given. The general case is analyzed in detail, providing orthogonality weight function, three-term recurrence relations for the monic orthogonal polynomial solutions, as well as explicit form of these monic orthogonal polynomial solutions, which are solutions of an admissible and potentially self-adjoint linear second-order partial differential equation of hypergeometric type.

Keywords Bivariate orthogonal polynomials · Symmetric orthogonal polynomials · Partial differential equations

Mathematics Subject Classification 33C50 · 33C65

Communicated by Majid Soleimani-damaneh.

✉ Iván Area
area@uvigo.gal

Yves Guemo Tefo
guemoyves2000@yahoo.fr

Rabia Aktaş
raktas@science.ankara.edu.tr

Esra Güldoğan Lekesiz
esra.guldogan@atilim.edu.tr

1 Universidade de Vigo, Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo, Campus As Lagoas-Ourense, 32004 Ourense, Spain

2 Department of Mathematics, Faculty of Science, University of Yaoundé I, Yaoundé, Cameroon

3 Department of Mathematics, Faculty of Science, Ankara University, 06100 Tandoğan, Ankara, Turkey

4 Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey

Published online: 12 July 2021

1 Introduction

Classical univariate orthogonal polynomials (of a continuous variable) can be introduced and characterized by different ways. One possible approach is to present them as solution of a second-order differential equation of hypergeometric type [29]

$$a(x)z''(x) + b(x)z'(x) + \mu_n z(x) = 0, \tag{1.1}$$

where $a(x)$ and $b(x)$ are polynomials of degree at most two and one, respectively, and μ_n is a constant, usually referred as eigenvalue. The hypergeometric property means that if $z(x)$ is a solution of (1.1), then any derivative of $z(x)$ is solution of an equation of the same type. The characterization problem has been analyzed by several authors [9]. As it is well known, the Favard theorem [31] links orthogonality and three-term recurrence relation which in the univariate situation reads as

$$x p_n(x) = r_n p_{n+1}(x) + s_n p_n(x) + t_n p_{n-1}(x), \quad t_n \neq 0.$$

Classical univariate orthogonal polynomials (of a continuous variable and orthogonal with respect a positive weight function) are essentially the Jacobi, Laguerre and Hermite polynomials, which can be expressed in terms of hypergeometric functions. Moreover, the differential equation (1.1) can be discretized in several forms, giving rise to classical orthogonal polynomials of a discrete variable and classical orthogonal polynomials on nonuniform lattices [29]. All of them can be expressed in terms of hypergeometric series [14].

There are also several approaches in the multivariate context. Some authors have focused the analysis on the partial differential equation [15], but it is possible to introduce multivariate analogues of classical orthogonal polynomials from another approaches [2,32–34]. A systematic analysis of the orthogonal polynomial solutions of some partial differential equations has been done by Suetin [30]. Let Π_n be the set of polynomials of total degree n in the multivariate case. There are three main concepts on the partial differential equation to be considered:

1. Admissible: It might exist a sequence of eigenvalues $\{\mu_n\}$ $(n = 0, 1, \dots)$ such that for each nonnegative integer n there are exactly $n + 1$ linearly independent solutions in Π_n and the equation has not non-trivial solutions in Π_k for $k < n$;
2. Hypergeometric: If $y(x_1, \dots, x_n)$ is solution of the partial differential equation, then

$$\frac{\partial^{r_1+r_2+\cdots+r_n}}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_n^{r_n}} y(x_1, \dots, x_n)$$

   is solution of an equation of the same type.
3. Potentially self-adjoint: From the partial differential equation, it is possible to introduce an operator $\mathcal{D}$. The operator (and therefore the equation) is potentially self-adjoint in a domain Ω if there exists a positive real function r in this domain such that the operator $r\mathcal{D}$ is self-adjoint in the domain Ω.

Potentially self-adjoint second-order partial differential equations have been deeply analyzed by Suetin [30] (see Chapter V). The concept of hypergeometric was introduced by Lyskova [17,18] with the name *basic class*. Admissible equations go back to the works of Krall and Sheffer [15]. In recent works [5], all these conditions have been imposed to the partial differential equations obtaining the algebraic and differential properties (as well as the weight functions) of orthogonal polynomial solutions of bivariate second-order linear partial differential equations which are admissible, potentially self-adjoint and of hypergeometric type. This analysis has been performed also for the discrete case [4] as well as for their q-analogues [6]. Furthermore, Lee [16] characterized centrally symmetric orthogonal polynomials satisfying an admissible partial differential equation of the form

$$\begin{aligned} &A\frac{\partial^2}{\partial x^2}u(x,y)+2B\frac{\partial^2}{\partial x\partial y}u(x,y)+C\frac{\partial^2}{\partial y^2}u(x,y)+D\frac{\partial}{\partial x}u(x,y)+E\frac{\partial}{\partial y}u(x,y)\\ &\quad=\left(ax^2+d_1x+e_1y+f_1\right)u_{xx}+\left(2axy+d_2x+e_2y+f_2\right)u_{xy}\\ &\qquad+\left(ay^2+d_3x+e_3y+f_3\right)u_{yy}+\left(gx+h_1\right)u_x+\left(gy+h_2\right)u_y=\mu_n u, \end{aligned}$$

where $\mu_n=an\,(n-1)+gn$. Lee [16] shows that these centrally symmetric orthogonal polynomials are either the product of Hermite polynomials which are solutions of the partial differential equation [15]

$$u_{xx}+u_{yy}-xu_x-yu_y=-nu,$$

or the circle polynomials which are solutions of the partial differential equation [15]

$$\left(x^2-1\right)u_{xx}+2xyu_{xy}+\left(y^2-1\right)u_{yy}+gxu_x+gyu_y=n\,(n+g-1)\,u.$$

These polynomials satisfy the property $u_n(-x,-y)=(-1)^n u_n(x,y)$.

Many special functions of mathematical physics, such as associated Legendre functions [8], Hermite functions [11,31], ultraspherical functions [11,31], appear as solutions of a regular or singular Sturm–Liouville problems. These functions satisfy a symmetry property as $\psi_n(-x)=(-1)^n\psi_n(x)$ and they have many applications in engineering and physics [8,28].

On the other hand, the very classical second-order linear differential equation (1.1) has been symmetrically extended [19–22] in the following way. Let $\psi_n(-x)=(-1)^n\psi_n(x)$ be a sequence of symmetric functions satisfying the equation

$$K(x)\psi_n''(x)+L(x)\psi_n'(x)+(\mu_n M(x)+Q(x)+\kappa_n O(x))\psi_n(x)=0, \tag{1.2}$$

where μ_n is a sequence of constants and

$$\kappa_n=\frac{1-(-1)^n}{2}. \tag{1.3}$$

Under certain conditions on $K(x)$, $L(x)$, $(M(x) > 0)$, $Q(x)$, and $O(x)$ it is possible to obtain the orthogonality weight function for the solutions of the latter equation.

In [20], for such an equation, a basic class of symmetric orthogonal polynomials with four free parameters whose special sub-cases are four main sequences of symmetric orthogonal polynomials such as the generalized ultraspherical polynomials [10,11], generalized Hermite polynomials [11,31] and two other sequences of symmetric polynomials, which are finitely orthogonal on $(-\infty, \infty)$, is introduced and its standard properties, such as orthogonality relation, three-term recurrence relation and so on, have been investigated. In [21,22], using the extended Sturm–Liouville theorem for symmetric functions, two basic classes of symmetric orthogonal functions with five parameters and six parameters are obtained for second-order differential equations in the special forms of (1.2). Furthermore, using a generalization of Sturm–Liouville problems in discrete spaces, a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters, which generalizes all classical discrete symmetric orthogonal polynomials, is introduced in [23,24]. For some other papers, we refer [3,25–27]. By the motivation of these papers in the univariate case, we consider now the bivariate case. In this direction, in bivariate case we consider a class of partial differential equations that shall be shown to have symmetric solutions, using the Sturm–Liouville theorem to prove their orthogonality.

The main aim of this work is to present a symmetric generalization of previous works obtaining a new class of partial differential equations having symmetric orthogonal solutions. For this class, conditions on the equation to be admissible are given and the general case is analyzed in detail. We would like to notice that, to the best of our knowledge, this type of partial differential equation has not been considered before in the literature, as providing orthogonal polynomial solutions of it. The interest of having these new orthogonal polynomial families is not only theoretical. Since they do satisfy a partial differential equation it might be possible to consider orthogonal polynomial expansions of related partial differential equations by considering, e.g., the so-called Navima algorithm [7,13], to get numerical approximations for the solutions to other partial differential equations.

The structure of this work is as follows. In Sect. 2, the new symmetric generalization to the bivariate case is presented, and orthogonality of the polynomial solutions is derived. Section 3 is devoted to analyze the conditions to have an admissible partial differential equation. Under these conditions several properties are obtained, namely: the three-term recurrence relations of the orthogonal polynomial solutions, the domain of orthogonality, as well as the explicit form of the monic orthogonal polynomial solutions, which are solutions of an admissible and potentially self-adjoint linear second-order partial differential equation of hypergeometric type.

2 Bivariate Symmetric Orthogonal Solutions

Let us consider the following second-order linear partial differential equation

$$K(x,y)\frac{\partial^2}{\partial x^2}u(x,y)+2L(x,y)\frac{\partial^2}{\partial x\partial y}u(x,y)+M(x,y)\frac{\partial^2}{\partial y^2}u(x,y)$$
$$+Q(x,y)\frac{\partial}{\partial x}u(x,y)+O(x,y)\frac{\partial}{\partial y}u(x,y)$$
$$+[\mu_n\varphi(x,y)+S(x,y)+\kappa_n T(x,y)]\,u(x,y)=0, \qquad (2.1)$$

where κ_n is defined in (1.3). We shall impose that for each $n = 0, 1, 2, \ldots$ the latter equation has symmetric polynomial solutions $u_n(x, y)$ of total degree n so that $u_n(-x, -y) = (-1)^n u_n(x, y)$. Then,

$$K(-x,-y)=K(x,y),\quad L(-x,-y)=L(x,y),\quad M(-x,-y)=M(x,y),$$
$$\varphi(-x,-y)=\varphi(x,y),\quad S(-x,-y)=S(x,y),\quad T(-x,-y)=T(x,y),$$
$$Q(-x,-y)=-Q(x,y),\quad O(-x,-y)=-O(x,y).$$

Following [30], let us define the linear operator

$$\mathcal{D}_s=K(x,y)\frac{\partial^2}{\partial x^2}+2L(x,y)\frac{\partial^2}{\partial x\partial y}+M(x,y)\frac{\partial^2}{\partial y^2}+Q(x,y)\frac{\partial}{\partial x}+O(x,y)\frac{\partial}{\partial y}. \qquad (2.2)$$

If the operator (2.2) is self-adjoint, then

$$Q(x,y)=\frac{\partial}{\partial x}K(x,y)+\frac{\partial}{\partial y}L(x,y),\qquad O(x,y)=\frac{\partial}{\partial x}L(x,y)+\frac{\partial}{\partial y}M(x,y).$$

Let us assume that (2.2) is not self-adjoint and there exists a function $\varrho(x, y)$ such that $\varrho\mathcal{D}_s$ is self-adjoint in a certain domain Ω. Then,

$$\varrho(x,y)Q(x,y)=\frac{\partial}{\partial x}[\varrho(x,y)K(x,y)]+\frac{\partial}{\partial y}[\varrho(x,y)L(x,y)],$$
$$\varrho(x,y)O(x,y)=\frac{\partial}{\partial x}[\varrho(x,y)L(x,y)]+\frac{\partial}{\partial y}[\varrho(x,y)M(x,y)].$$

Hence,

$$K(x,y)\frac{\partial}{\partial x}\varrho(x,y)+L(x,y)\frac{\partial}{\partial y}\varrho(x,y)=\varrho(x,y)\left(Q(x,y)-\frac{\partial}{\partial x}K(x,y)-\frac{\partial}{\partial y}L(x,y)\right),$$
$$L(x,y)\frac{\partial}{\partial x}\varrho(x,y)+M(x,y)\frac{\partial}{\partial y}\varrho(x,y)=\varrho(x,y)\left(O(x,y)-\frac{\partial}{\partial x}L(x,y)-\frac{\partial}{\partial y}M(x,y)\right).$$

Let

$$\zeta(x,y)=K(x,y)M(x,y)-L^2(x,y),\quad \eta(x,y)=M(x,y)\phi(x,y)-L(x,y)\theta(x,y),$$
$$\tau(x,y)=K(x,y)\theta(x,y)-L(x,y)\phi(x,y),$$

where

$$\phi(x,y) = Q(x,y) - \frac{\partial}{\partial x}K(x,y) - \frac{\partial}{\partial y}L(x,y) = \frac{K(x,y)}{\varrho(x,y)}\frac{\partial}{\partial x}\varrho(x,y) + \frac{L(x,y)}{\varrho(x,y)}\frac{\partial}{\partial y}\varrho(x,y),$$
$$\theta(x,y) = O(x,y) - \frac{\partial}{\partial x}L(x,y) - \frac{\partial}{\partial y}M(x,y) = \frac{L(x,y)}{\varrho(x,y)}\frac{\partial}{\partial x}\varrho(x,y) + \frac{M(x,y)}{\varrho(x,y)}\frac{\partial}{\partial y}\varrho(x,y).$$

Under these conditions (2.1) is potentially self-adjoint in a domain Ω if

$$\frac{\partial}{\partial x}\left(\frac{\tau(x,y)}{\zeta(x,y)}\right) = \frac{\partial}{\partial y}\left(\frac{\eta(x,y)}{\zeta(x,y)}\right) \tag{2.3}$$

holds true. Since

$$\frac{1}{\varrho(x,y)}\frac{\partial}{\partial x}\varrho(x,y) = \frac{\eta(x,y)}{\zeta(x,y)}, \quad \frac{1}{\varrho(x,y)}\frac{\partial}{\partial y}\varrho(x,y) = \frac{\tau(x,y)}{\zeta(x,y)},$$

we obtain

$$\varrho(x,y) = \exp\left[\int_{y_0}^{y}\frac{\tau(x,y)}{\zeta(x,y)}dy + \int_{x_0}^{x}\frac{\eta(x,y_0)}{\zeta(x,y_0)}dx + c\right]. \tag{2.4}$$

Since K, L, and M are even polynomials and Q, O are odd polynomials, we have

$$\begin{cases}\phi(-x,-y) = -\phi(x,y), \quad \theta(-x,-y) = -\theta(x,y), \quad \zeta(-x,-y) = \zeta(x,y),\\ \eta(-x,-y) = -\eta(x,y), \quad \tau(-x,-y) = -\tau(x,y).\end{cases} \tag{2.5}$$

Thus, we have the following symmetry relation for the function $\varrho(x,y)$ defined in (2.4):

$$\varrho(-x,-y) = \varrho(x,y).$$

Under these conditions, it is possible to write (2.1) in self-adjoint form as

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\varrho(x,y)K(x,y)\frac{\partial}{\partial x}u(x,y) + \varrho(x,y)L(x,y)\frac{\partial}{\partial y}u(x,y)\right]\\ &+\frac{\partial}{\partial y}\left[\varrho(x,y)L(x,y)\frac{\partial}{\partial x}u(x,y) + \varrho(x,y)M(x,y)\frac{\partial}{\partial y}u(x,y)\right]\\ &+\varrho(x,y)\left[\mu_n\varphi(x,y) + S(x,y) + \kappa_n T(x,y)\right]u(x,y) = 0.\end{aligned} \tag{2.6}$$

Let us write (2.6) for $u(x,y) = u_n(x,y)$:

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\varrho(x,y)K(x,y)\frac{\partial}{\partial x}u_n(x,y) + \varrho(x,y)L(x,y)\frac{\partial}{\partial y}u_n(x,y)\right]\\ &+\frac{\partial}{\partial y}\left[\varrho(x,y)L(x,y)\frac{\partial}{\partial x}u_n(x,y) + \varrho(x,y)M(x,y)\frac{\partial}{\partial y}u_n(x,y)\right]\\ &+\varrho(x,y)\left[\mu_n\varphi(x,y) + S(x,y) + \kappa_n T(x,y)\right]u_n(x,y) = 0,\end{aligned} \tag{2.7}$$

and for $u(x, y) = u_m(x, y)$:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\varrho(x,y)K(x,y)\frac{\partial}{\partial x}u_m(x,y)+\varrho(x,y)L(x,y)\frac{\partial}{\partial y}u_m(x,y)\right]\\
&+\frac{\partial}{\partial y}\left[\varrho(x,y)L(x,y)\frac{\partial}{\partial x}u_m(x,y)+\varrho(x,y)M(x,y)\frac{\partial}{\partial y}u_m(x,y)\right]\\
&+\varrho(x,y)\left[\mu_m\varphi(x,y)+S(x,y)+\kappa_m T(x,y)\right]u_m(x,y)=0. \qquad (2.8)
\end{aligned}$$

If we multiply (2.7) by $u_m(x, y)$ and (2.8) by $u_n(x, y)$, and then substract them, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x}\left\{\varrho(x,y)K(x,y)\left[u_m(x,y)\frac{\partial}{\partial x}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial x}u_m(x,y)\right]\right.\\
&\left.+\varrho(x,y)L(x,y)\left[u_m(x,y)\frac{\partial}{\partial y}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial y}u_m(x,y)\right]\right\}\\
&+\frac{\partial}{\partial y}\left\{\varrho(x,y)L(x,y)\left[u_m(x,y)\frac{\partial}{\partial x}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial x}u_m(x,y)\right]\right.\\
&\left.+\varrho(x,y)M(x,y)\left[u_m(x,y)\frac{\partial}{\partial y}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial y}u_m(x,y)\right]\right\}\\
&+\varrho(x,y)(\mu_n-\mu_m)\varphi(x,y)u_n(x,y)u_m(x,y)\\
&-\frac{(-1)^m-(-1)^n}{2}\varrho(x,y)T(x,y)u_n(x,y)u_m(x,y)=0. \qquad (2.9)
\end{aligned}$$

If we take integral on both sides, we have

$$\begin{aligned}
&\iint_\Omega \frac{\partial}{\partial x}\left\{\varrho(x,y)K(x,y)\left[u_m(x,y)\frac{\partial}{\partial x}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial x}u_m(x,y)\right]\right.\\
&\left.+\varrho(x,y)L(x,y)\left[u_m(x,y)\frac{\partial}{\partial y}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial y}u_m(x,y)\right]\right\}dxdy\\
&+\iint_\Omega \frac{\partial}{\partial y}\left\{\varrho(x,y)L(x,y)\left[u_m(x,y)\frac{\partial}{\partial x}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial x}u_m(x,y)\right]\right.\\
&\left.+\varrho(x,y)M(x,y)\left[u_m(x,y)\frac{\partial}{\partial y}u_n(x,y)-u_n(x,y)\frac{\partial}{\partial y}u_m(x,y)\right]\right\}dxdy\\
&-\frac{(-1)^m-(-1)^n}{2}\iint_\Omega \varrho(x,y)T(x,y)u_n(x,y)u_m(x,y)dxdy\\
&=(\mu_m-\mu_n)\iint_\Omega \varrho(x,y)\varphi(x,y)u_n(x,y)u_m(x,y)dxdy. \qquad (2.10)
\end{aligned}$$

Remark 2.1 Based on the analysis of Suetin [30] on potentially self-adjoint second-order partial differential equations, in the case $\varphi(x, y) = 1$, $S(x, y) = T(x, y) = 0$ of the partial differential equation (2.1), if the main differential operator is admissible

and potentially self-adjoint in a simply connected domain Ω, there exists a weight function $\varrho(x, y)$ of this operator in the domain Ω and the solutions of the equation are orthogonal with respect to this weight function on the domain Ω. For the general equation (2.1), the calculations above show that if the equation is potentially self-adjoint, the function $\varrho(x, y)$ is determined up to a constant factor and if the left side of (2.10) vanishes, for $\mu_n \neq \mu_m$ the eigenfunctions of the Eq. (2.1) are orthogonal with respect to the weight function $w(x, y) = \varrho(x, y)\varphi(x, y)$, which is positive (for $\varphi(x, y) > 0$) and even function on Ω.

3 The Class of Admissible Partial Differential Equations

Let us assume that K, L and M are polynomials of total degree 4, Q and O are polynomials of total degree 3, and φ and T are polynomials of total degree 2, S is identically zero, such that the symmetry conditions (2.5) hold true:

$$\begin{cases} K(x, y) = k_0x^4 + k_1x^3y + k_2x^2y^2 + k_3xy^3 + k_4y^4 + k_5x^2 + k_6xy + k_7y^2 + k_8, \\ L(x, y) = l_0x^4 + l_1x^3y + l_2x^2y^2 + l_3xy^3 + l_4y^4 + l_5x^2 + l_6xy + l_7y^2 + l_8, \\ M(x, y) = m_0x^4 + m_1x^3y + m_2x^2y^2 + m_3xy^3 + m_4y^4 + m_5x^2 + m_6xy + m_7y^2 + m_8, \\ Q(x, y) = q_0x^3 + q_1x^2y + q_2xy^2 + q_3y^3 + q_4x + q_5y, \\ O(x, y) = o_0x^3 + o_1x^2y + o_2xy^2 + o_3y^3 + o_4x + o_5y, \\ \varphi(x, y) = \varphi_0x^2 + \varphi_1xy + \varphi_2y^2 + \varphi_3, \quad S(x, y) = 0, \\ T(x, y) = t_0x^2 + t_1xy + t_2y^2 + t_3. \end{cases}$$

In this section, we shall give conditions on the coefficients of the above polynomials to (2.1) be an admissible partial differential equation, that is, a partial differential equation having $n + 1$ linearly independent polynomial solutions for each nonnegative integer n.

Let $\mathbf{x} = (x, y) \in \mathbf{R}^2$. For each nonnegative integer n, let $\mathbf{x}^n$ be the column vector of the monomials $x^{n-k}y^k$, whose elements are arranged in graded lexicographical order [12, p. 32]

$$\mathbf{x}^n = \left(x^{n-k}y^k\right), \quad k = 0, 1, \ldots, n. \tag{3.1}$$

In this way,

$$\mathbf{x}^1 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{x}^2 = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}, \ldots.$$

Let Π_n^2 be the space of all polynomials in two variables $\mathbf{x} = (x, y)$ of total degree at most n. Then, any element of Π_n^2 can be expressed as finite sums of terms of the form $c_kx^{n-k}y^k$, where c_k are real constants.

If we apply the partial differential operator to $\mathbf{x}^n$ defined in (3.1), we obtain

$$\mu_n = -\frac{1}{\varphi_0}\left(nq_0 + n\left(n-1\right)k_0 + \kappa_nt_0\right) \tag{3.2}$$

under the restrictions

$$\begin{aligned}
&l_0 = m_0 = o_0 = m_1 = k_3 = q_3 = k_4 = l_4 = 0,\\
&k_0 = k_1 = k_2 = l_1 = l_2 = l_3 = m_2 = m_3 = m_4,\\
&q_0 = q_1 = q_2 = o_1 = o_2 = o_3,\\
&t_0 = t_1 = t_2 \ , \ \varphi_0 = \varphi_1 = \varphi_2.
\end{aligned}$$

If we impose that the partial differential equation has a monic solution, that is

$$\mathbb{P}_1(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbb{P}_2(x, y) = \begin{pmatrix} x^2 + \varpi_1 \\ xy + \varpi_2 \\ y^2 + \varpi_3 \end{pmatrix},$$
$$\mathbb{P}_3(x, y) = \begin{pmatrix} x^3 + \varpi_4 x \\ x^2 y + \varpi_5 x + \varpi_6 y \\ xy^2 + \varpi_7 x + \varpi_8 y \\ y^3 + \varpi_9 y \end{pmatrix}, \ldots,$$

then the following conditions appear:

$$\begin{aligned}
&o_4 = q_5 = 0, \quad m_6 = m_5, \quad l_7 = l_5, \quad k_7 = k_6, \quad q_4 = o_5 = \frac{q_0\varphi_3}{\varphi_0},\\
&k_8 = \frac{\varphi_3(k_5\varphi_0 - k_0\varphi_3)}{\varphi_0^2}, \quad l_8 = \frac{l_5\varphi_3}{\varphi_0}, \quad k_6 = k_5 - \frac{k_0\varphi_3}{\varphi_0},\\
&l_6 = l_5 + \frac{k_0\varphi_3}{\varphi_0}, \quad m_7 = m_5 + \frac{k_0\varphi_3}{\varphi_0}, \quad m_8 = \frac{m_5\varphi_3}{\varphi_0}, \quad t_3 = \frac{\varphi_3 t_0}{\varphi_0}.
\end{aligned}$$

Without loss of generality we can assume $\varphi_0 = 1$ and we obtain

$$\begin{cases}
K(x, y) = \left(k_0\left(x^2 - \varphi_3\right) + k_5\right)\left(\varphi_3 + x^2 + xy + y^2\right),\\
L(x, y) = \left(\varphi_3 + x^2 + xy + y^2\right)\left(k_0 xy + l_5\right),\\
M(x, y) = \left(k_0 y^2 + m_5\right)\left(\varphi_3 + x^2 + xy + y^2\right),\\
Q(x, y) = q_0 x\left(\varphi_3 + x^2 + xy + y^2\right),\\
O(x, y) = q_0 y\left(\varphi_3 + x^2 + xy + y^2\right),\\
\varphi(x, y) = x^2 + xy + y^2 + \varphi_3, \quad S(x, y) = 0,\\
T(x, y) = t_0\left(\varphi_3 + x^2 + xy + y^2\right).
\end{cases} \tag{3.3}$$

Under these conditions and μ_n defined in (3.2) the equation (2.1) is admissible, i.e., for each nonnegative integer n the equation has precisely $n + 1$ linearly independent solutions in the form of polynomials of total degree n and has no non-trivial solutions in the set of polynomials whose total degree is less than n. Hence,

$$\begin{aligned}
\zeta(x,y) &= \left(\varphi_3 + x^2 + xy + y^2\right)^2 \\
&\quad \times \left(-2k_0l_5xy - k_0\varphi_3\left(k_0y^2 + m_5\right) + k_0m_5x^2 + k_5\left(k_0y^2 + m_5\right) - l_5^2\right), \\
\eta(x,y) &= \left(\varphi_3 + x^2 + xy + y^2\right)\left(\varphi_3\left(k_0^2y^2(2x+y) + m_5(-k_0x + k_0y + q_0x)\right)\right. \\
&\quad + l_5y\left(\varphi_3(3k_0 - q_0) + k_0\left(7x^2 + 5xy + 3y^2\right) - q_0\left(x^2 + xy + y^2\right)\right) \\
&\quad + m_5x\left(x^2(q_0 - 5k_0) + xy(q_0 - 4k_0) + y^2(q_0 - 3k_0)\right) \\
&\quad \left. - k_5(2x+y)\left(k_0y^2 + m_5\right) + l_5^2(2x+y)\right), \\
\tau(x,y) &= \left(\varphi_3 + x^2 + xy + y^2\right)\left((k_0xy + l_5)\left(\varphi_3(k_0x - k_0y - q_0x)\right.\right. \\
&\quad \left. + x(5k_0 - q_0)\left(x^2 + xy + y^2\right) + k_5(2x+y) + l_5(x+2y)\right) - \left(k_0\left(x^2 - \varphi_3\right) + k_5\right) \\
&\quad \left. \times \left(\varphi_3y(3k_0 - q_0) + y(5k_0 - q_0)\left(x^2 + xy + y^2\right) + l_5(2x+y) + m_5(x+2y)\right)\right).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\frac{\tau(x,y)}{\zeta(x,y)} &= -\frac{(3k_0 - q_0)(k_0xy + l_5)(m_5x - l_5y)}{\left(k_0y^2 + m_5\right)\left(2k_0l_5xy + k_0\varphi_3\left(k_0y^2 + m_5\right) - k_0m_5x^2 - k_5\left(k_0y^2 + m_5\right) + l_5^2\right)} \\
&\quad + \frac{y(q_0 - 3k_0)}{k_0y^2 + m_5} - \frac{x + 2y}{\varphi_3 + x^2 + xy + y^2},
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{\eta(x,y)}{\zeta(x,y)} &= -\frac{(3k_0 - q_0)(l_5y - m_5x)}{2k_0l_5xy + k_0\varphi_3\left(k_0y^2 + m_5\right) - k_0m_5x^2 - k_5\left(k_0y^2 + m_5\right) + l_5^2} \\
&\quad - \frac{2x + y}{\varphi_3 + x^2 + xy + y^2}.
\end{aligned}$$

Under these conditions the equation becomes potentially self-adjoint since (2.3) holds true. Moreover,

$$\varrho(x,y) = \frac{\left(2k_0l_5xy + k_0\varphi_3\left(k_0y^2 + m_5\right) - k_0m_5x^2 - k_5\left(k_0y^2 + m_5\right) + l_5^2\right)^{\frac{1}{2}\left(\frac{q_0}{k_0} - 3\right)}}{\varphi_3 + x^2 + xy + y^2},$$

so that

$$\begin{aligned}
w(x,y) &= \varphi(x,y)\varrho(x,y) \\
&= \left(2k_0l_5xy + k_0\varphi_3\left(k_0y^2 + m_5\right) - k_0m_5x^2 - k_5\left(k_0y^2 + m_5\right) + l_5^2\right)^{\frac{1}{2}\left(\frac{q_0}{k_0} - 3\right)}.
\end{aligned}$$

The orthogonality domain is defined by $\zeta(x,y) \geq 0$. For instance, if we fix $m_5 = 1$, $k_5 = 4k_0\varphi_3$, $k_0 = -1$, $\varphi_3 = -1$ (so that $k_5 = 4$), $q_0 = -7$ and $l_5 = 0$, then the domain Ω is defined by

$$\frac{x^2}{3} + y^2 \leq 1,$$

the orthogonality weight function reduces to

$$w(x, y) = (x^2 + 3y^2 - 3)^2,$$

and the polynomials become orthogonal with any polynomial of lower total degree, as it has been shown.

3.1 Three-Term Recurrence Relations for the Monic Orthogonal Polynomial Solutions

Let us assume that the partial differential equation has a monic symmetric polynomial solution of the form

$$\mathbb{P}_n = \mathbf{x}^n + \widehat{G}_{n,n-2}\mathbf{x}^{n-2} + \cdots,$$

where $\mathbf{x}^n$ is defined in (3.1). Let

$$L_{n,1}\mathbf{x}^{n+1} = x\,\mathbf{x}^n, \quad L_{n,2}\mathbf{x}^{n+1} = y\,\mathbf{x}^n,$$

be matrices of size $(n+1)\times(n+2)$. Then,

$$L_{n,1} = \begin{pmatrix} 1 & \bigcirc & 0 \\ & \ddots & \vdots \\ \bigcirc & 1 & 0 \end{pmatrix} \text{ and } \quad L_{n,2} = \begin{pmatrix} 0 & 1 & \bigcirc \\ \vdots & & \ddots \\ 0 & \bigcirc & 1 \end{pmatrix}. \tag{3.4}$$

Hence,

$$x^2\,\mathbf{x}^n = L_{n,1}L_{n+1,1}\mathbf{x}^{n+2}, \quad y^2\,\mathbf{x}^n = L_{n,2}L_{n+1,2}\mathbf{x}^{n+2}, \quad L_{n,2}L_{n+1,1} = L_{n,1}L_{n+1,2},$$

and for $j = 1, 2$,

$$L_{n,j}\,L_{n,j}^{\mathrm{T}} = I_{n+1},$$

where I_{n+1} stands for the identity matrix of size $n+1$.

Moreover, let

$$\mathbb{E}_{n,1} = \begin{pmatrix} n & & \bigcirc \\ & n-1 & \\ & & \ddots \\ & \bigcirc & & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \text{ and } \quad \mathbb{E}_{n,2} = \begin{pmatrix} 0 & \cdots & & 0 \\ 1 & & & \bigcirc \\ & 2 & & \\ & & \ddots & \\ \bigcirc & & & n \end{pmatrix},$$

be matrices of size $(n+1)\times n$. For $n \geq 1$,

$$\partial_x\mathbf{x}^n = \mathbb{E}_{n,1}\,\mathbf{x}^{n-1}, \quad \partial_y\mathbf{x}^n = \mathbb{E}_{n,2}\,\mathbf{x}^{n-1}.$$

If we apply the second-order partial differential equation (2.1) to $\mathbf{x}^n$, we obtain that the matrix $\widehat{G}_{n,n-2} \in \mathcal{M}(n+1, n-1)$ is tridiagonal

$$\widehat{G}_{n,n-2} = \begin{pmatrix} \varpi_{1,1}(n) & 0 & & 0 & \cdots & 0 \\ \varpi_{2,1}(n) & \varpi_{2,2}(n) & & 0 & \cdots & 0 \\ \varpi_{3,1}(n) & \varpi_{3,2}(n) & \varpi_{3,3}(n) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & \\ \vdots & & & \varpi_{n-1,n-3}(n) & \varpi_{n-1,n-2}(n) & \varpi_{n-1,n-1}(n) \\ 0 & 0 & \cdots & 0 & \varpi_{n,n-2}(n) & \varpi_{n,n-1}(n) \\ 0 & 0 & 0 & \cdots & 0 & \varpi_{n+1,n-1}(n) \end{pmatrix} \tag{3.5}$$

and

$$\varpi_{i,j}(n) = \frac{1}{(2n-3)k_0 + q_0} \times \begin{cases} \binom{n-i+1}{2}(k_5 - k_0\varphi_3), & i = j, \\ (i-1)(n-i+1)l_5, & j = i-1, \\ \binom{i-1}{2}m_5, & j = i-2. \end{cases} \tag{3.6}$$

As it is well known [12], for $n \geq 0$, there exist unique matrices $A_{n,j}$ of the size $(n+1) \times (n+2)$, $B_{n,j}$ of the size $(n+1) \times (n+1)$, and $C_{n,j}$ of the size $(n+1) \times n$, such that

$$x_j \mathbb{P}_n = A_{n,j}\mathbb{P}_{n+1} + B_{n,j}\mathbb{P}_n + C_{n,j}\mathbb{P}_{n-1}, \quad j = 1, 2, \tag{3.7}$$

with the initial conditions $\mathbb{P}_{-1} = 0$ and $\mathbb{P}_0 = 1$ where we have denoted $x_1 = x$ and $x_2 = y$. In the symmetric case $B_{n,j}$ is zero. From [5, Theorem 4.2] in this symmetric case, we have:

Corollary 3.1 *In the monic case, the explicit expressions of the matrices $A_{n,j}$ and $C_{n,j}$ ($j = 1, 2$), that appear in (3.7) in terms of the values of $\widehat{G}_{n,n-2}$, are given by*

$$\begin{cases} A_{n,j} = L_{n,j}, \quad n \geq 0, \\ C_{1,j} = -L_{1,j}\widehat{G}_{2,0}, \\ C_{n,j} = \widehat{G}_{n,n-2}L_{n-2,j} - L_{n,j}\widehat{G}_{n+1,n-1}, \quad n \geq 2, \end{cases}$$

where the matrices $L_{n,j}$ have been introduced in (3.4), and the coefficients of the matrix $\widehat{G}_{n,n-2}$ defined in (3.5) are given in (3.6).

The matrix $C_{n,1} \in \mathcal{M}(n+1,n)$ is tridiagonal

$$C_{n,1} = \frac{1}{((2n-1)k_0+q_0)((2n-3)k_0+q_0)} \times \begin{pmatrix} \varphi^1_{1,1}(n) & 0 & & 0 & \cdots & 0 \\ \varphi^1_{2,1}(n) & \varphi^1_{2,2}(n) & & 0 & \cdots & 0 \\ \varphi^1_{3,1}(n) & \varphi^1_{3,2}(n) & \varphi^1_{3,3}(n) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & \\ \vdots & & & \varphi^1_{n-1,n-2}(n) & \varphi^1_{n,n-1}(n) & \varphi^1_{n,n}(n) \\ 0 & 0 & \cdots & 0 & \varphi^1_{n+1,n-1}(n) & \varphi^1_{n+1,n}(n) \end{pmatrix},$$

where

$$\varphi^1_{i,j}(n) = \begin{cases} (n-i+1)((n+i-3)k_0+q_0)(k_0\varphi_3-k_5), & i=j, \\ -j((2j-3)k_0+q_0)l_5, & j=i-1, \\ j(j+1)k_0m_5, & j=i-2. \end{cases}$$

The matrix $C_{n,2} \in \mathcal{M}(n+1,n)$ is also tridiagonal

$$C_{n,2} = \frac{1}{((2n-1)k_0+q_0)((2n-3)k_0+q_0)} \times \begin{pmatrix} \varphi^2_{1,1}(n) & \varphi^2_{1,2}(n) & 0 & 0 & \cdots & 0 \\ \varphi^2_{2,1}(n) & \varphi^2_{2,2}(n) & \varphi^2_{2,3}(n) & 0 & \cdots & 0 \\ 0 & \varphi^2_{3,2}(n) & \varphi^2_{3,3}(n) & \varphi^2_{3,4}(n) & \cdots & 0 \\ 0 & \ddots & \ddots & & \varphi^2_{n-1,n-1}(n) & \varphi^2_{n-1,n}(n) \\ \vdots & & & 0 & \varphi^2_{n,n-1}(n) & \varphi^2_{n,n}(n) \\ 0 & 0 & \cdots & 0 & 0 & \varphi^2_{n+1,n}(n) \end{pmatrix},$$

where

$$\varphi^2_{i,j}(n) = \begin{cases} -(n-i+1)((2(n-i)-1)k_0+q_0)l_5, & i=j, \\ -(i-1)((2n-i-1)k_0+q_0)m_5, & j=i-1, \\ -(n-i+1)(n-i)k_0(k_0\varphi_3-k_5), & j=i+1. \end{cases}$$

3.2 Explicit Form of the Monic Orthogonal Polynomial Solutions

Next, we obtain the explicit form of the monic polynomial solutions of the partial differential equation (2.1) for the specific form of the polynomials (3.3).

Proposition 3.2 *Assuming that the polynomials of* (2.1) *are given by* (3.3)*, then the monic polynomial solutions of the partial differential equation* (2.1) *can be explicitly given as:*

$$\begin{aligned} P_{n,k}(x,y) = &\sum_{s=0}^{k}\binom{k}{s}\sum_{r=[(s+1)/2]}^{[(n-k+s)/2]}\binom{n-k}{2r-s}x^{n-k+s-2r}y^{k-s}\frac{\prod_{j=0}^{r-1-[s/2]}(2j+1)}{\prod_{j=0}^{r-1}((2(n-j)-3)k_0+q_0)} \\ &\times\sum_{j=0}^{[s/2]}\frac{2^{2j-[s/2]}\,(s-2j-\kappa_s)!}{([s/2]-j)!}\binom{s}{\kappa_s+2j} \\ &\times(k_5-k_0\varphi_3)^{r-s+[s/2]-j}(r-[(s+1)/2]-j+1)_j\,l_5^{2j+\kappa_s}m_5^{[s/2]-j}, \end{aligned} \tag{3.8}$$

for $0\le k\le n$*, where* κ_n *is defined in* (1.3)*,* $(\lambda)_n=\lambda(\lambda+1)\cdots(\lambda+n-1)$ *denotes the Pochhammer symbol for a nonnegative integer n, with* $(\lambda)_0=1$*, and we use the convention* $\prod_{j=1}^{0}a_j=1$*.*

Proof For the specific form of the polynomials (3.3), plug into (2.1) the expansion (3.8) to obtain that all the coefficients vanish. □

3.3 Partial Differential Equation of Hypergeometric Type

Moreover, the partial derivatives of $P_{n,k}$ also satisfy an equation of the same type. More precisely:

Proposition 3.3 *Let* $P_{n,k}(x,y)$ *be the monic orthogonal polynomials defined in* (3.8) *solutions to the second-order partial differential equation* (2.1) *for the specific values of the polynomials given in* (3.3)*. Then, the partial derivatives of* $P_{n,k}(x,y)$

$$v_{n,k}^{(r,s)}(x,y)=\frac{\partial^r}{\partial x^r}\frac{\partial^s}{\partial y^s}P_{n,k}(x,y)$$

are solution of an equation of the same type as (2.1)

$$\begin{aligned} &K(x,y)\frac{\partial^2}{\partial x^2}v_{n,k}^{(r,s)}(x,y)+2L(x,y)\frac{\partial^2}{\partial x\partial y}v_{n,k}^{(r,s)}(x,y)+M(x,y)\frac{\partial^2}{\partial y^2}v_{n,k}^{(r,s)}(x,y) \\ &+Q_{r,s}(x,y)\frac{\partial}{\partial x}v_{n,k}^{(r,s)}(x,y)+O_{r,s}(x,y)\frac{\partial}{\partial y}v_{n,k}^{(r,s)}(x,y) \\ &+\left[\mu_n^{(r,s)}\varphi(x,y)+\kappa_nT(x,y)\right]v_{n,k}^{(r,s)}(x,y)=0, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} Q_{r,s}(x,y)&=(2(r+s)k_0+q_0)(x^3+x^2y+xy^2+\varphi_3), \\ O_{r,s}(x,y)&=(2(r+s)k_0+q_0)(x^2y+xy^2+y^3+\varphi_3), \\ \mu_n^{(r,s)}&=-(n-r-s)(k_0(n+r+s-1)+q_0)-\kappa_nt_0. \end{aligned}$$

Proof The result follows by direct substitution of the partial derivatives of the explicit expression (3.8) into (3.9) and equate the coefficients. □

Remark 3.4 Since the polynomials K, L and M in the equation for the partial derivatives (3.9) do not depend on r and s, then the polynomial $\zeta^{(r,s)}(x, y)$ is exactly the same as for the initial equation. Moreover,

$$\begin{aligned}
\eta^{(r,s)}(x, y) = &\left(\varphi_3 + x^2 + xy + y^2\right)\left(k_0^2\varphi_3 y^2(2x + y)\right.\\
&+k_0\left(xy^2\left(-2k_5 + l_5(-2r - 2s + 5)\right.\right.\\
&\left.+m_5(2r + 2s - 3)\right) - y^3\left(k_5 + l_5(2r + 2s - 3)\right)\\
&+\varphi_3 y\left(l_5(-2r - 2s + 3) + m_5\right)\\
&+x^2 y\left(l_5(-2r - 2s + 7) + 2m_5(r + s - 2)\right) + m_5\varphi_3 x(2r + 2s - 1)\\
&\left.+m_5 x^3(2r + 2s - 5)\right) + (2x + y)\left(l_5^2 - k_5 m_5\right)\\
&\left.+q_0\left(\varphi_3 + x^2 + xy + y^2\right)(m_5 x - l_5 y)\right),\\
\tau^{(r,s)}(x, y) = &\left(\varphi_3 + x^2 + xy + y^2\right)\left(\left(k_0\left(x^2 - \varphi_3\right) + k_5\right)(k_0 y\left(\varphi_3(2r + 2s - 3)\right.\right.\\
&\left.+(2r + 2s - 5)\left(x^2 + xy + y^2\right)\right) + l_5(-(2x + y)) - m_5(x + 2y)\\
&+q_0 y\left(\varphi_3 + x^2 + xy + y^2\right)) - (k_0 xy + l_5)(k_0\varphi_3(x(2r + 2s - 1) + y)\\
&+k_0 x(2r + 2s - 5)\left(x^2 + xy + y^2\right) - k_5(2x + y) - l_5(x + 2y)\\
&\left.+q_0 x\left(\varphi_3 + x^2 + xy + y^2\right)\right)).
\end{aligned}$$

Since

$$\frac{\partial}{\partial x}\left(\frac{\tau^{(r,s)}(x, y)}{\zeta(x, y)}\right) = \frac{\partial}{\partial y}\left(\frac{\eta^{(r,s)}(x, y)}{\zeta(x, y)}\right),$$

the equation for the partial derivatives (3.9) is again potentially self-adjoint, as well as hypergeometric since the partial derivatives of any order of any solution of (3.9) are solution of an equation of the same type.

Acknowledgements The authors thank the referees for their valuable comments which improved a preliminary version of this work. The work of R.A. has been partially supported by TUBITAK Research Grant Proj. No. 120F140. The work of I.A. has been partially supported by the Agencia Estatal de Investigación (AEI) of Spain under Grant MTM2016-75140-P, cofinanced by the European Community fund FEDER. Funding for open access charge: Universidade de Vigo/CISUG.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Andrews, G.E., Askey, R.: Classical orthogonal polynomials. In: Brezinski, C., et al. (eds.) Polynômes Orthogonaux et Applications. Lecture Notes in Mathematics, vol. 1171, pp. 36–62. Springer, Berlin (1985)
2. Area, I.: Hypergeometric multivariate orthogonal polynomials. In: Foupouagnigni, M., Koepf, W. (eds.) Orthogonal Polynomials, Tutorials, Schools, and Workshops in the Mathematical Sciences. Springer Nature, Switzerland AG (2020)
3. Area, I., Masjed-Jamei, M.: A class of symmetric q-orthogonal polynomials with four free parameters. Bull. Sci. Math. **141**(8), 785–801 (2017)
4. Area, I., Godoy, E., Rodal, J.: On a class of bivariate second-order linear partial difference equations and their monic orthogonal polynomial solutions. J. Math. Anal. Appl. **389**, 165–178 (2012)
5. Area, I., Godoy, E., Ronveaux, A., Zarzo, A.: Bivariate second-order linear partial differential equations and orthogonal polynomial solutions. J. Math. Anal. Appl. **387**, 1188–1208 (2012)
6. Area, I., Atakishiyev, N., Godoy, E., Rodal, J.: Linear partial q-difference equations on q-linear lattices and their bivariate q-orthogonal polynomial solutions. Appl. Math. Comput. **223**, 520–536 (2013)
7. Area, I., Godoy, E., Rodal, J., Ronveaux, A., Zarzo, A.: Bivariate Krawtchouk polynomials: Inversion and connection problems with the NAVIMA algorithm. J. Comput. Appl. Math. **284**, 50–57 (2015)
8. Arfken, G.B., Weber, H.J., Harris, F.E.: Mathematical Methods for Physicists. Seventh edition. Academic Press, San Diego, CA (2012)
9. Atakishiyev, N.M., Rahman, M., Suslov, S.K.: On classical orthogonal polynomials. Constr. Approx. **11**(2), 181–226 (1995)
10. Charris, J.A., Ismail, M.E.H.: On sieved orthogonal polynomials II: Random walk polynomials. Can. J. Math. **38**, 397–415 (1986)
11. Chihara, T.S.: Introduction to Orthogonal Polynomials. Gordon and Breach, New York (1978)
12. Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables, Encyclopedia of Mathematics and Its Applications, vol. 81. Cambridge University Press, Cambridge (2001)
13. Godoy, E., Ronveaux, A., Zarzo, A., Area, I.: Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Continuous case. J. Comput. Appl. Math. **84**, 257–275 (2007)
14. Koekoek, R., Lesky, P.A., Swarttouw, R.F.: Hypergeometric Orthogonal Polynomials and Their q-Analogues. Springer, Berlin (2010)
15. Krall, H.L., Sheffer, I.M.: Orthogonal polynomials in two variables. Ann. Mat. Pura Appl. **4**, 325–376 (1967)
16. Lee, J.K.: Centrally symmetric orthogonal polynomials in two variables. Comm. Korean Math. Soc. **12**(3), 645–653 (1997)
17. Lyskova, A.S.: Orthogonal polynomials in several variables. Sov. Math. Dokl. **43**, 264–268 (1991)
18. Lyskova, A.S.: On some properties of orthogonal polynomials in several variables. Russ. Math. Surv. **52**, 840–841 (1997)
19. Masjed-Jamei, M.: A generalization of classical symmetric orthogonal functions using a symmetric generalization of Sturm-Liouville problems. Integral Transf. Spec. Funct. **18**, 871–883 (2007)
20. Masjed-Jamei, M.: A basic class of symmetric orthogonal polynomials using the extended Sturm–Liouville theorem for symmetric functions. J. Math. Anal. Appl. **325**, 753–775 (2007)
21. Masjed-Jamei, M.: A basic class of symmetric orthogonal functions using the extended Sturm–Liouville theorem for symmetric functions. J. Comput. Appl. Math. **216**(1), 128–143 (2008)
22. Masjed-Jamei, M.: A basic class of symmetric orthogonal functions with six free parameters. J. Comput. Appl. Math. **234**(1), 283–296 (2010)
23. Masjed-Jamei, M., Area, I.: A symmetric generalization of Sturm-Liouville problems in discrete spaces. J. Differ. Equ. Appl. **19**(9), 1544–1562 (2013)
24. Masjed-Jamei, M., Area, I.: A basic class of symmetric orthogonal polynomials of a discrete variable. J. Math. Anal. Appl. **399**(1), 291–305 (2013)
25. Masjed-Jamei, M., Koepf, W.: On incomplete symmetric orthogonal polynomials of Jacobi type. Integral Transf. Spec. Funct. **21**(9), 655–662 (2010)
26. Masjed-Jamei, M., Koepf, W.: On incomplete symmetric orthogonal polynomials of Laguerre type. Appl. Anal. **90**(3–4), 769–775 (2011)
27. Masjed-Jamei, M., Moalemi, Z., Saad, N.: Incomplete symmetric orthogonal polynomials of finite type generated by a generalized Sturm-Liouville theorem. J. Math. Phys. **61**(2), 023501 (2020)

28. Nikiforov, A.F., Uvarov, V.B.: Special Functions of Mathematical Physics. Birkhauser, Basel-Boston (1988)
29. Nikiforov, A.F., Suslov, S.K.., Uvarov, V.B.: Classical orthogonal polynomials of a discrete variable. In: Springer Series in Computational Physics. Springer, Berlin (1991)
30. Suetin, P.K.: Orthogonal Polynomials in Two Variables. Gordon and Breach Science Publishers, Amsterdam (1999)
31. Szegö, G.: Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., vol. XXIII. AMS, Providence (1978)
32. Tratnik, M.V.: Multivariable Meixner, Krawtchouk, and Meixner–Pollaczek polynomials. J. Math. Phys. **30**, 2740–2749 (1989)
33. Tratnik, M.V.: Some multivariable orthogonal polynomials of the Askey tableau—continuous families. J. Math. Phys. **32**, 2065–2073 (1991)
34. Tratnik, M.V.: Some multivariable orthogonal polynomials of the Askey tableau—discrete families. J. Math. Phys. **32**, 2337–2342 (1991)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.